Probability and Inferential Statistics

Lecture SS 16

Densities of Continuous Test Statistics

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Continuous Random Variable and its Density

Definition 1. Let $X : \Omega \to \mathbb{R}$ be a real-valued random variable on $(\Omega, \mathscr{A}, P)$, and F_X its distribution function. Then X is called *continuous*, if there is a Riemann-integrable function $f_X : \mathbb{R} \to \mathbb{R}$, such that

$$F_X(x) := \int_{-\infty}^{x} f_X(t)\, dt, \quad x \in \mathbb{R}.$$

A function f_X satisfying this equation is called a *density* of X.

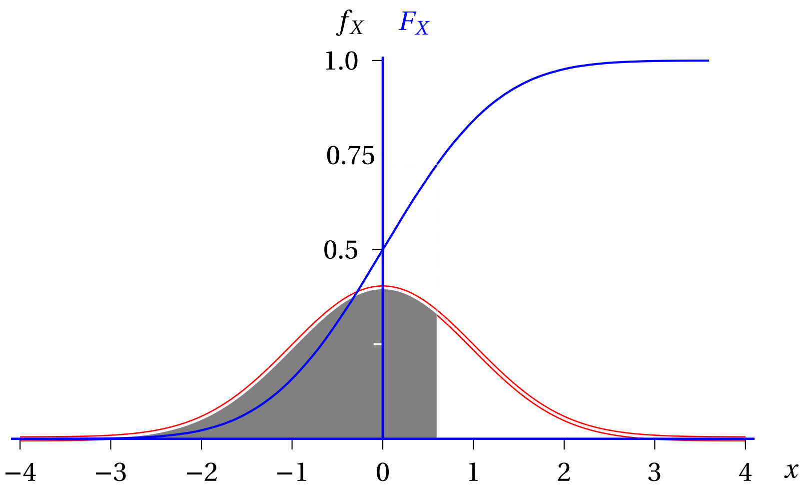
According to this definition, the area under the density $f(x)$ — this is what is computed by the Riemann integral — is the probability that X takes on a value less than or equal to x.

A well-known density of a random variable X is the density of a *normal distribution*, which is defined by

$$f_X(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \cdot exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right),$$

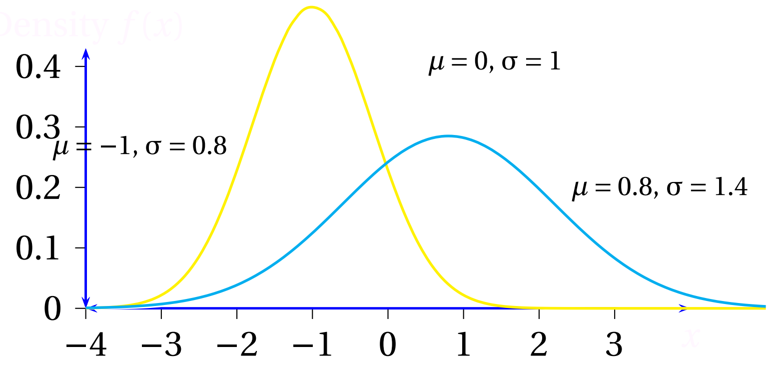
where μ and σ^2 are real numbers, the *expectation* of X and its *variance*, respectively.

Density and Distribution Function of the Normal Distribution



Density and distribution function of a normal distribution

Densities of Three Normal Distributions



R-Programs for the Normal Distribution

Description

- Density, distribution function, quantile function and random generation for the normal distribution with mean equal to *mean* and standard deviation equal to *sd*.

Usage

- *dnorm*(x, *mean*=0, *sd*=1, *log* = FALSE) Density
- *pnorm*(q, *mean*=0, *sd*=1, lower.tail = TRUE, *log.p* = FALSE) Distribution function
- *qnorm*(p, *mean*=0, *sd*=1, lower.tail = TRUE, *log.p* = FALSE) Quantile function
- *rnorm*(n, *mean*=0, *sd*=1) Data sample of size n of a normally distributed sample

Arguments

- x, q vector of quantiles.
- p vector of probabilities.
- n number of observations. If *length*(n) > 1, the length is taken to be the number required.
- *mean* vector of means.
- *sd* vector of standard deviations.
- *log*, *log.p* logical; if TRUE, probabilities p are given as *log*(p).
- lower.tail logical; if TRUE (default), probabilities are $P(X \leq x)$, otherwise, $P(X > x)$.

Table of the Standard Normal Distribution

Table I

Cumulative normal probabilities

z	*F*(*z*)	*z*	*F*(*z*)	*z*	*F*(*z*)	*z*	*F*(*z*)
.00	.5000000	.21	.5831662	.42	.6627573	.63	.7356527
.01	.5039894	.22	.5870604	.43	.6664022	.64	.7389137
.02	.5079783	.23	.5909541	.44	.6700314	.65	.7421539
.03	.5119665	.24	.5948349	.45	.6736448	.66	.7453731
.04	.5159534	.25	.5987063	.46	.6772419	.67	.7485711
.05	.5199388	.26	.6025681	.47	.6808225	.68	.7517478
.06	.5239222	.27	.6064199	.48	.6843863	.69	.7549029
.07	.5279032	.28	.6102612	.49	.6879331	.70	.7580363
.08	.5318814	.29	.6140919	.50	.6914625	.71	.7611479
.09	.5358564	.30	.6179114	.51	.6949743	.72	.7642375
.10	.5398278	.31	.6217195	.52	.6984682	.73	.7673049
.11	.5437953	.32	.6255158	.53	.7019440	.74	.7703500
.12	.5477584	.33	.6293000	.54	.7054015	.75	.7733726
.13	.5517168	.34	.6330717	.55	.7088403	.76	.7763727
.14	.5556700	.35	.6368307	.56	.7122603	.77	.7793501
.15	.5596177	.36	.6405764	.57	.7156612	.78	.7823046
.16	.5635595	.37	.6443088	.58	.7190427	.79	.7852361
.17	.5674949	.38	.6480273	.59	.7224047	.80	.7881446
.18	.5714237	.39	.6517317	.60	.7257469	.81	.7910299
.19	.5753454	.40	.6554217	.61	.7290691	.82	.7938919
.20	.5792597	.41	.6590970	.62	.7323711	.83	.7967306

From: Hays, W. L. (1994). *Statistics*. Fort Worth, TX: Harcourt Brace.

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Table of the Standard Normal Distribution continued 1

Table I (continued)

z	*F*(*z*)	*z*	*F*(*z*)	*z*	*F*(*z*)	*z*	*F*(*z*)
.84	.7995458	1.32	.9065825	1.79	.9632730	2.26	.9880894
.85	.8023375	1.33	.9082409	1.80	.9640697	2.27	.9883962
.86	.8051055	1.34	.9098773	1.81	.9648521	2.28	.9886962
.87	.8078498	1.35	.9114920	1.82	.9656205	2.29	.9889893
.88	.8105703	1.36	.9130850	1.83	.9663750	2.30	.9892759
.89	.8132671	1.37	.9146565	1.84	.9671159	2.31	.9895559
.90	.8159399	1.38	.9162067	1.85	.9678432	2.32	.9898296
.91	.8185887	1.39	.9177356	1.86	.9685572	2.33	.9900969
.92	.8212136	1.40	.9192433	1.87	.9692581	2.34	.9903581
.93	.8238145	1.41	.9207302	1.88	.9699460	2.35	.9906133
.94	.8263912	1.42	.9221962	1.89	.9706210	2.36	.9908625
.95	.8289439	1.43	.9236415	1.90	.9712834	2.37	.9911060
.96	.8314724	1.44	.9250663	1.91	.9719334	2.38	.9913437
.97	.8339768	1.45	.9264707	1.92	.9725711	2.39	.9915758
.98	.8364569	1.46	.9278550	1.93	.9731966	2.40	.9918025
.99	.8389129	1.47	.9292191	1.94	.9738102	2.41	.9920237
1.00	.8413447	1.48	.9305634	1.95	.9744119	2.42	.9922397
1.01	.8437524	1.49	.9318879	1.96	.9750021	2.43	.9924506
1.02	.8461358	1.50	.9331928	1.97	.9755808	2.44	.9926564
1.03	.8484950	1.51	.9344783	1.98	.9761482	2.45	.9928572
1.04	.8508300	1.52	.9357445	1.99	.9767045	2.46	.9930531
1.05	.8531409	1.53	.9369916	2.00	.9772499	2.47	.9932443

From: Hays, W. L. (1994). *Statistics*. Fort Worth, TX: Harcourt Brace.

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Table of the Standard Normal Distribution continued 1

1.06	.8554277	1.54	.9382198	2.01	.9777844	2.48	.9934309
1.07	.8576903	1.55	.9394292	2.02	.9783083	2.49	.9936128
1.08	.8599289	1.56	.9406201	2.03	.9788217	2.50	.9937903
1.09	.8621434	1.57	.9417924	2.04	.9793248	2.51	.9939634
1.10	.8643339	1.58	.9429466	2.05	.9798178	2.52	.9941323
1.11	.8665005	1.59	.9440826	2.06	.9803007	2.53	.9942969
1.12	.8686431	1.60	.9452007	2.07	.9807738	2.54	.9944574
1.13	.8707619	1.61	.9463011	2.08	.9812372	2.55	.9946139
1.14	.8728568	1.62	.9473839	2.09	.9816911	2.56	.9947664
1.15	.8749281	1.63	.9484493	2.10	.9821356	2.57	.9949151
1.16	.8769756	1.64	.9494974	2.11	.9825708	2.58	.9950600
1.17	.8789995	1.65	.9505285	2.12	.9829970	2.59	.9952012
1.18	.8809999	1.66	.9515428	2.13	.9834142	2.60	.9953388
1.19	.8829768	1.67	.9525403	2.14	.9838226	2.70	.9965330
1.20	.8849303	1.68	.9535213	2.15	.9842224	2.80	.9974449
1.21	.8868606	1.69	.9544860	2.16	.9846137	2.90	.9981342
1.22	.8887676	1.70	.9554345	2.17	.9849966	3.00	.9986501
1.23	.8906514	1.71	.9563671	2.18	.9853713	3.20	.9993129
1.24	.8925123	1.72	.9572838	2.19	.9857379	3.40	.9996631
1.25	.8943502	1.73	.9581849	2.20	.9860966	3.60	.9998409
1.26	.8961653	1.74	.9590705	2.21	.9864474	3.80	.9999277
1.27	.8979577	1.75	.9599408	2.22	.9867906	4.00	.9999683
1.28	.8997274	1.76	.9607961	2.23	.9871263	4.50	.9999966
1.29	.9014747	1.77	.9616364	2.24	.9874545	5.00	.9999997
1.30	.9031995	1.78	.9624620	2.25	.9877755	5.50	.9999999
1.31	.9049021						

From: Hays, W. L. (1994). *Statistics*. Fort Worth, TX: Harcourt Brace.

Density of the Central χ^2-Distribution

In the following definition, $\Gamma\colon \mathbb{R} \to \mathbb{R}$ denotes the *gamma function* defined by

$$\Gamma(a) := \int_0^\infty t^{a-1} e^{-t}\, dt, \quad \forall a \in \mathbb{R}, \quad a > 0. \tag{1}$$

Note that,

$$\Gamma(a) = (a-1)\cdot\Gamma(a-1), \quad \text{for } a > 1. \tag{2}$$

Furthermore,

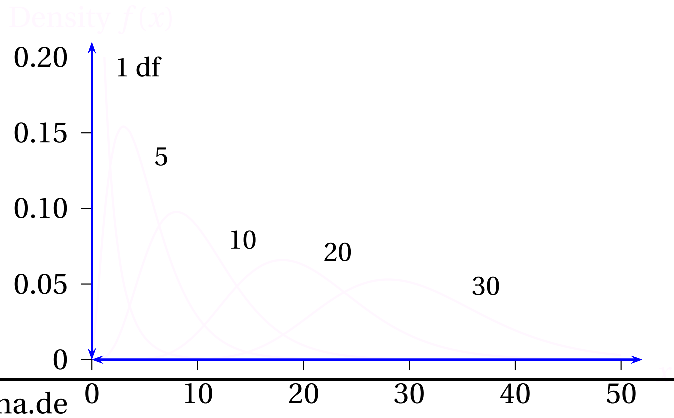
$$\Gamma(a) = (a-1)! \quad \text{for } a \in \mathbb{N}, \quad \text{and} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{3}$$

Definition 2 (Central χ^2-Distribution)**.**
Let $n \in \mathbb{N}$. A continuous nonnegative random variable $X\colon (\Omega, \mathscr{A}, P) \to (\mathbb{R}, \mathscr{B})$ has a *central χ^2-distribution with n degrees of freedom*, abbreviated $X \sim \chi^2_n$, if X has a density satisfying

$$f_X(x) = \begin{cases} \dfrac{x^{n/2-1}\cdot e^{-x/2}}{2^{n/2}\cdot\Gamma(n/2)}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases} \quad \forall x \in \mathbb{R}. \tag{4}$$

If $X \sim \chi^2_n$, then $E(X) = n$ and $Var(X) = 2n$.

χ^2-Distributions With 1, 5, 10, 20 and 30 Degrees of Freedom



Normal Distribution and χ^2-Distribution

Normal and χ^2-distributions are related to each other as follows:

If $X_1, \ldots, X_n$ are i. i.d. random variables with standard normal distribution, then

$$X = \sum_{i=1}^{n} X_i^2 \tag{5}$$

has a central χ^2-distribution with n degrees of freedom.

R-Program for the χ^2-Distribution

Description

- Density, distribution function, quantile function and random generation for the χ^2 distribution with *df* degrees of freedom and optional non-centrality parameter *ncp*.

Usage

- *dchisq*(x, *df*, *ncp*=0, *log* = *FALSE*)   Density
- *pchisq*(q, *df*, *ncp*=0, *lower.tail* = *TRUE*, *log.p* = *FALSE*)   Distribution function
- *qchisq*(p, *df*, *ncp*=0, *lower.tail* = *TRUE*, *log.p* = *FALSE*) Quantile function
- *rchisq*(n, *df*, *ncp*=0)   Data sample of size n of a χ^2-distributed sample

Arguments

- x, q vector of quantiles.
- p vector of probabilities.
- n number of observations. If *length*(n) > 1, the length is taken to be the number required.
- *df* degrees of freedom (non-negative, maybe non-integer).
- *ncp* non-centrality parameter (non-negative);
- *log*, *log.p* logical; if *TRUE*, probabilities p are given as $log(p)$.
- *lower.tail* logical; if *TRUE* (default), probabilities are $P(X \leq x)$, otherwise, $P(X > x)$.

χ^2-Distribution Table

Tabelle B.2: P-Quantile $\chi^2(f;p)$ der χ^2-Verteilung (kritische Werte des χ^2-Tests) mit f Freiheitsgraden

f	P							
	0,01	0,025	0,05	0,1	0,9	0,95	0,975	0,99
1	$1571 \cdot 10^{-7}$	$9821 \cdot 10^{-7}$	$3932 \cdot 10^{-6}$	0,01579	2,706	3,841	5,024	6,635
2	0,02010	0,05064	0,1026	0,2107	4,605	5,991	7,378	9,210
3	0,1148	0,2158	0,3518	0,5844	6,251	7,815	9,348	11,34
4	0,2971	0,4844	0,7107	1,064	7,779	9,488	11,14	13,28
5	0,5543	0,8312	1,145	1,610	9,236	11,07	12,83	15,09
6	0,8721	1,237	1,635	2,204	10,64	12,59	14,45	16,81
7	1,239	1,690	2,167	2,833	12,02	14,07	16,01	18,48
8	1,646	2,180	2,733	3,490	13,36	15,51	17,53	20,09
9	2,088	2,700	3,325	4,168	14,68	16,92	19,02	21,67
10	2,558	3,247	3,940	4,865	15,99	18,21	20,48	23,21
11	3,053	3,816	4,575	5,578	17,28	19,68	21,92	24,72
12	3,571	4,404	5,226	6,304	18,55	21,03	23,34	26,22
13	4,107	5,009	5,892	7,042	19,81	22,36	24,74	27,69
14	4,660	5,629	6,571	7,790	21,06	23,68	26,12	29,14
15	5,229	6,262	7,261	8,547	22,31	25,00	27,49	30,58
16	5,812	6,908	7,962	9,312	23,54	26,30	28,85	32,00
17	6,408	7,564	8,672	10,09	24,77	27,59	30,19	33,41
18	7,015	8,231	9,390	10,86	25,99	28,87	31,53	34,81
19	7,633	8,907	10,12	11,65	27,20	30,14	32,85	36,19
20	8,260	9,591	10,85	12,44	28,41	31,41	34,17	37,57

From: Rasch, D. & Kubinger, K. D. (2006), *Statistik für das Psychologiestudium*. Heidelberg: Elsevier.

t-Distribution

In the following definition we again use the gamma function Γ defined by Equation (1).

Definition 3.
Let $n \in \mathbb{N}$. A continuous random variable $X : (\Omega, \mathscr{A}, P) \to (\mathbb{R}, \mathscr{B})$ has a *central t-distribution with n degrees of freedom*, denoted $X \sim t_n$, if X has a density satisfying

$$f_X(x) \;=\; \frac{\Gamma\big((n+1)/2\big)}{\sqrt{n\,\pi}\cdot\Gamma\,(n/2)}\left(1+\frac{x^2}{n}\right)^{-(n+1)/2}, \quad \forall x \in \mathbb{R}. \tag{6}$$

If $X \sim t_n$ and $n > 1$, then $E(X) = 0$, and if $n > 2$, then $Var(X) = n/(n-2)$.

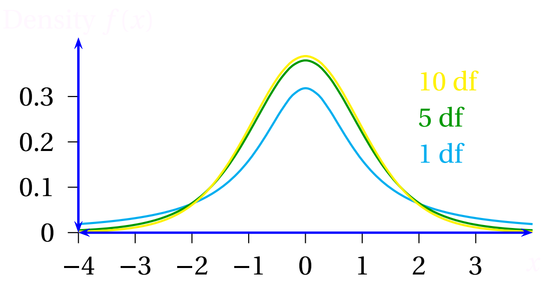
If $Z \sim \mathscr{N}_{0,1}$, $Y \sim \chi^2_n$, and Z and Y are independent, then

$$X \;:=\; \frac{Z}{\sqrt{\frac{Y}{n}}} \tag{7}$$

has a t-distribution with n degrees of freedom.

Central t-Distribution With $df = 1, 5, 10$

The following figure displays the densities of three t-distributions with 1, 5, and 10 degrees of freedom, respectively.



R-Programs for the t-Distribution

Description

- Density, distribution function, quantile function and random generation for the t distribution with *df* degrees of freedom (and optional non-centrality parameter *ncp*).

Usage

- *dt*(*x*, *df*, *ncp*, *log* = *FALSE*) Density function
- *pt*(*q*, *df*, *ncp*, *lower.tail* = *TRUE*, *log.p* = *FALSE*) Distribution function
- *qt*(*p*, *df*, *ncp*, *lower.tail* = *TRUE*, *log.p* = *FALSE*) Quantile function
- *rt*(*n*, *df*, *ncp*) Data sample of size n of a t-distributed sample

Arguments

- x, q vector of quantiles.
- p vector of probabilities.
- n number of observations. If *length*(n) > 1, the length is taken to be the number required.
- df degrees of freedom (> 0, maybe non-integer). *df* = *Inf* is allowed. For *qt* only values of at least one are currently supported.
- *ncp* non-centrality parameter delta; currently except for *rt*( ), only for *abs*(*ncp*) ≤ 37.62. If omitted, use the central t distribution.
- *log*, *log.p* logical; if *TRUE*, probabilities p are given as $log(p)$.
- lower.tail logical; if *TRUE* (default), probabilities are $P(X \leq x)$, otherwise, $P(X > x)$.

t-Distribution Table

Tabelle B.1: P-Quantile der t-Verteilung mit f Freiheitsgraden (für $f = \infty$, P-Quantile der Standardnormalverteilung)

f	P								
	0,6	0,7	0,8	0,85	0,9	0,95	0,975	0,99	0,995
1	0,3249	0,7265	1,3764	1,9626	3,0777	6,3138	12,7062	31,8205	63,6567
2	0,2887	0,6172	1,0607	1,3862	1,8856	2,9200	4,3027	6,9646	9,9248
3	0,2767	0,5844	0,9785	1,2498	1,6377	2,3534	3,1824	4,5407	5,8409
4	0,2707	0,5686	0,9410	1,1896	1,5332	2,1318	2,7764	3,7469	4,6041
5	0,2672	0,5594	0,9195	1,1558	1,4759	2,0150	2,5706	3,3649	4,0321
6	0,2648	0,5534	0,9057	1,1342	1,4398	1,9432	2,4469	3,1427	3,7074
7	0,2632	0,5491	0,8960	1,1192	1,4149	1,8946	2,3646	2,9980	3,4995
8	0,2619	0,5459	0,8889	1,1081	1,3968	1,8595	2,3060	2,8965	3,3554
9	0,2610	0,5435	0,8834	1,0997	1,3830	1,8331	2,2622	2,8214	3,2498
10	0,2602	0,5415	0,8791	1,0931	1,3722	1,8125	2,2281	2,7638	3,1693
11	0,2596	0,5399	0,8755	1,0877	1,3634	1,7959	2,2010	2,7181	3,1058
12	0,2590	0,5386	0,8726	1,0832	1,3562	1,7823	2,1788	2,6810	3,0545
13	0,2586	0,5375	0,8702	1,0795	1,3502	1,7709	2,1604	2,6503	3,0123
14	0,2582	0,5366	0,8681	1,0763	1,3450	1,7613	2,1448	2,6245	2,9768
15	0,2579	0,5357	0,8662	1,0735	1,3406	1,7531	2,1314	2,6025	2,9467
16	0,2576	0,5350	0,8647	1,0711	1,3368	1,7459	2,1199	2,5835	2,9208
17	0,2573	0,5344	0,8633	1,0690	1,3334	1,7396	2,1098	2,5669	2,8982
18	0,2571	0,5338	0,8620	1,0672	1,3304	1,7341	2,1009	2,5524	2,8784
19	0,2569	0,5333	0,8610	1,0655	1,3277	1,7291	2,0930	2,5395	2,8609
20	0,2567	0,5329	0,8600	1,0640	1,3253	1,7247	2,0860	2,5280	2,8453

From: Rasch, D. & Kubinger, K. D. (2006), *Statistik für das Psychologiestudium.* Heidelberg: Elsevier.

F-Distribution

In the following definition we again use the gamma function Γ defined by Equation (1).

Definition 4 (Central F-Distribution)**.**
Let $m, n \in \mathbb{N}$ and let $X\colon (\Omega, \mathscr{A}, P) \to (\mathbb{R}, \mathscr{B})$ be a continuous nonnegative random variable. Then we say that X has a central *F-distribution with m and n degrees of freedom*, abbreviated $X \sim F_{m,n}$, if it has a density satisfying

$$f_X(x) \;=\; \begin{cases} \dfrac{\Gamma\big((m+n)/2\big)\cdot m^{m/2}\cdot n^{n/2}\cdot x^{m/2-1}}{\Gamma(m/2)\cdot\Gamma(n/2)\cdot(n+mx)^{(m+n)/2}}\,, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases} \qquad \forall x \in \mathbb{R}. \tag{8}$$

If $X \sim F_{m,n}$, then, for $n \geq 3$, the expectation of X is $E(X) = n/(n-2)$. For $n \leq 2$, the expectation does not exist. If $n \geq 5$ the variance of X is

$$Var(X) \;=\; \frac{2n^2\cdot(m+n-2)}{m\cdot(n-2)^2\cdot(n-4)}\,. \tag{9}$$

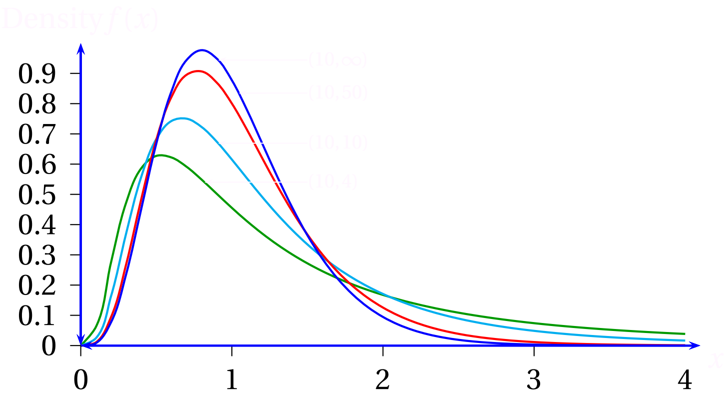
For $n \leq 4$, the variance of X does not exist.

If $Z \sim \chi^2_m$ and $Y \sim \chi^2_n$ are independent, then

$$X := \frac{Z/m}{Y/n} \sim F_{m,n}. \tag{10}$$

Some Densities of F-Distributions

The following figure displays the densities of four F-distributions.



R-Programs for the F-Distribution

Description

- Density, distribution function, quantile function and random generation for the F distribution with df_1 and df_2 degrees of freedom (and optional non-centrality parameter *ncp*).

Usage

- $df(x, df_1, df_2, ncp, log = FALSE)$ Density
- $pf(q, df_1, df_2, ncp, lower.tail = TRUE, log.p = FALSE)$ Distribution function
- $qf(p, df_1, df_2, ncp, lower.tail = TRUE, log.p = FALSE)$ Quantile function
- $rf(n, df_1, df_2, ncp)$ Data sample of size n of a F-distributed sample

Arguments

- x, q vector of quantiles.
- p vector of probabilities.
- n number of observations. If $length(n) > 1$, the length is taken to be the number required.
- df_1, df_2 degrees of freedom. *Inf* is allowed.
- *ncp* non-centrality parameter. If omitted the central F is assumed.
- *log*, *log.p* logical; if *TRUE*, probabilities p are given as $log(p)$.
- *lower.tail* logical; if *TRUE* (default), probabilities are $P(X \leq x)$, otherwise, $P(X > x)$.

F-Distribution Table

Tabelle B.3: 95%-Quantile der F-Verteilung mit f_1 und f_2 Freiheitsgraden

f_2 \ f_1	1	2	3	4	5	6	7	8	9
1	161,4	199,5	215,7	224,6	230,2	234,0	236,8	238,9	240,5
2	18,51	19,00	19,16	19,25	19,30	19,33	19,35	19,37	19,38
3	10,13	9,55	9,28	9,12	9,01	8,94	8,89	8,85	8,81
4	7,71	6,94	6,59	6,39	6,26	6,16	6,09	6,04	6,00
5	6,61	5,79	5,41	5,19	5,05	4,95	4,88	4,82	4,77
6	5,99	5,14	4,76	4,53	4,39	4,28	4,21	4,15	4,10
7	5,59	4,74	4,35	4,12	3,97	3,87	3,79	3,73	3,68
8	5,32	4,46	4,07	3,84	3,69	3,58	3,50	3,44	3,39
9	5,12	4,26	3,86	3,63	3,48	3,37	3,29	3,23	3,18
10	4,96	4,10	3,71	3,48	3,33	3,22	3,14	3,07	3,02
11	4,84	3,98	3,59	3,36	3,20	3,09	3,01	2,95	2,90
12	4,75	3,89	3,49	3,27	3,11	3,00	2,91	2,85	2,80
13	4,67	3,81	3,41	3,18	3,03	2,92	2,83	2,77	2,71
14	4,60	3,74	3,34	3,11	2,96	2,85	2,76	2,70	2,65
15	4,54	3,68	3,29	3,06	2,90	2,79	2,71	2,64	2,59
16	4,49	3,63	3,24	3,01	2,85	2,74	2,66	2,59	2,54
17	4,45	3,59	3,20	2,96	2,81	2,70	2,61	2,55	2,49
18	4,41	3,55	3,16	2,93	2,77	2,66	2,58	2,51	2,46
19	4,38	3,52	3,13	2,90	2,74	2,63	2,54	2,48	2,42
20	4,35	3,49	3,10	2,87	2,71	2,60	2,51	2,45	2,39

From: Rasch, D. & Kubinger, K. D. (2006), *Statistik für das Psychologiestudium.* Heidelberg: Elsevier.