Probability and Fundamentals of Inferential Statistics

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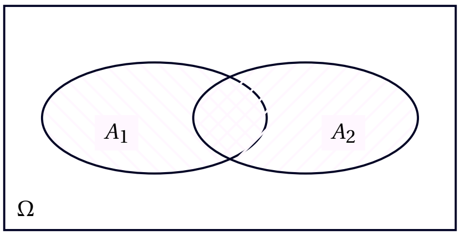
April 14, 2016

Definition of a Conditional Probability

Let $(\Omega, \mathscr{A}, P)$ be a probability space, $A_1, A_2 \in \mathscr{A}$, and $P(A_1) > 0$. Then

$$P(A_2 \mid A_1) := \frac{P(A_1 \cap A_2)}{P(A_1)}$$

is called the *conditional probability of* A_2 *given* A_1.



Multiplication Rule

If $P(A_1) > 0$, then:

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2 \mid A_1)$$

If $P(A_1 \cap A_2) > 0$, then:

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P[(A_1 \cap A_2) \cap A_3] \\ &= P(A_1 \cap A_2) \cdot P(A_3 \mid A_1 \cap A_2) \\ &= P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3 \mid A_1 \cap A_2) \end{aligned}$$

If $P(A_1 \cap \ldots \cap A_{n-1}) > 0$, then:

$$P(A_1 \cap \ldots \cap A_n) = P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3 \mid A_1 \cap A_2) \cdot \ldots \cdot P(A_n \mid A_1 \cap \ldots \cap A_{n-1})$$

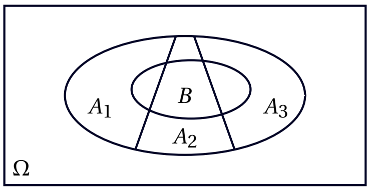
Theorem of Total Probability and Bayes Theorem

If
(a) the events $A_1, \ldots, A_n$ are pairwise disjoint,
(b) $B \subset A_1 \cup \ldots \cup A_n$, and
(c) $P(A_i) > 0$, for all $i = 1, \ldots, n$,
then:

$$\begin{aligned} P(B) &= P(B \cap A_1) + \ldots + P(B \cap A_n) \qquad \text{(total probability)} \\ &= P(B \,|\, A_1) \cdot P(A_1) + \ldots + P(B \,|\, A_n) \cdot P(A_n) \end{aligned}$$

and

$$P(A_i \,|\, B) = \frac{P(B \,|\, A_i) \cdot P(A_i)}{P(B \,|\, A_1) \cdot P(A_1) + \ldots + P(B \,|\, A_n) \cdot P(A_n)} \qquad \text{(Bayes)}$$



Independence of Events

(i) Two events A_1, A_2 are called *(stochastically) independent*, if

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2).$$

(ii) The events $A_1, \ldots, A_n$ are called *(stochastically) independent*, if

$$P(A_{i_1} \cap \ldots \cap A_{i_m}) = P(A_{i_1}) \cdot \ldots \cdot P(A_{i_m})$$
$$\text{for all } \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}.$$

Independence of the events $A_1, \ldots, A_n$ implies their pairwise independence. Pairwise independence of the events $A_1, \ldots, A_n$ does not imply independence of the events $A_1, \ldots, A_n$.

Independence of the three events A_1, A_2, A_3 implies

$$P(A_i \cap A_j) = P(A_i) \cdot P(A_j), \quad \text{for all } i, j = 1, 2, 3, \quad i \neq j,$$

and

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3).$$

Independence of Sets of Events

Definition 1. Let $(\Omega, \mathscr{A}, P)$ be a probability space. The sets of events $\mathscr{E}_1, \ldots, \mathscr{E}_n \subset \mathscr{A}$ are called *(stochastically) independent,* if all n-tupels $(A_1, \ldots, A_n) \in \mathscr{E}_1 \times \ldots \times \mathscr{E}_n$ are independent.

Note that this definition also applies to σ-algebras $\mathscr{A}_1, \ldots, \mathscr{A}_n \subset \mathscr{A}$.

Conditional-Probability Measure

Let $(\Omega, \mathscr{A}, P)$ be a probability space. If $B \in \mathscr{A}$ and $P(B) > 0$, then the function $P^B \colon \mathscr{A} \to [0, 1]$ defined by

$$P^B(A) = P(A \mid B) \qquad \text{for all } A \in \mathscr{A} \tag{1}$$

is a probability measure. It is called the B-conditional probability measure on $(\Omega, \mathscr{A})$.

Example: Joe and Ann with probabilities for elementary events

Table 1: Joe and Ann with probabilities for elementary events

Elements of Ω (Unit, Treatment, Success)	$P(\{\omega\})$
(*Joe*, *no*, −)	.09
(*Joe*, *no*, +)	.21
(*Joe*, *yes*, −)	.04
(*Joe*, *yes*, +)	.16
(*Ann*, *no*, −)	.24
(*Ann*, *no*, +)	.06
(*Ann*, *yes*, −)	.16
(*Ann*, *yes*, +)	.04

Example: Joe and Ann With Some Conditional Probabilities

- Which is the conditional probability of success given Joe is sampled and treated?
- Which is the conditional probability of success given Joe is sampled and not treated?
- Which is the conditional probability of success given the sampled person is treated?
- Which is the conditional probability of success given the sampled person is not treated?