# Lecture
# „Item Response Theory“

## The two parameter logistic Birnbaum model

29. November 2016

# 2-PL-Birnbaum model

- Model equation

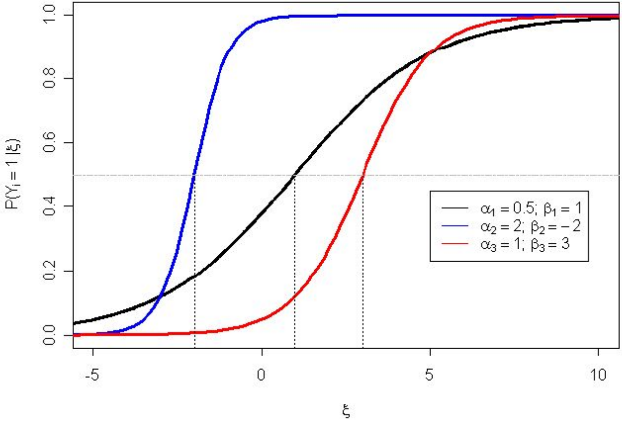
$$P(Y_i = 1 \mid U) = P(Y_i = 1 \mid \xi) = \frac{\exp\left[\alpha_i \cdot \; (\xi - \beta_i)\right]}{1 + \exp\left[\alpha_i \cdot \; (\xi - \beta_i)\right]}$$

- Local stochastic independence

$$P(Y_i = 1 \mid U, Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_m) = P(Y_i = 1 \mid U)$$

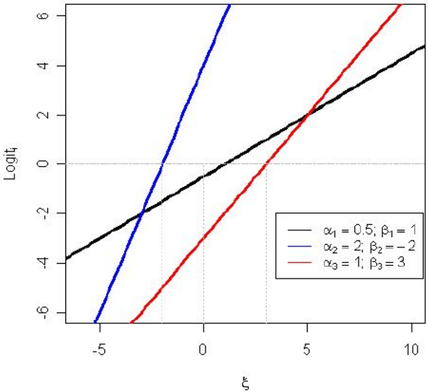
# 2-PL-Birnbaum model

- Item characteristic curves



# 2-PL-Birnbaum model

- Logits are linear functions of $\xi$ and of each other




$\Rightarrow$ The 2-PL-Birnbaum model is analog to the model of $\tau$-congeneric variables

$$\ln\left(\frac{P(Y_i = 1 \mid \xi)}{P(Y_i = 0 \mid \xi)}\right) = \alpha_i \cdot (\xi - \beta_i)$$

# 2-PL-Birnbaum model

- Meaning of the model parameters:

1. latent variable $\xi$ (e.g., ability, ...)
2. Item parameter:

   $\beta_i$: Item difficulty (same interpretation as in the Rasch model)

   $\alpha_i$: Item discrimination

# 2-PL-Birnbaum model

- Item information function: If we specify $\alpha_1 \equiv 1$ and :

$$P_i(\xi) \;:=\; \frac{\exp\left[\alpha_i \cdot (\xi - \beta_i)\right]}{1 + \exp\left[\alpha_i \cdot (\xi - \beta_i)\right]}, \qquad i = 1, \ldots, m,$$

  then the information function

$$I_{Y_i}(\xi) := \frac{[P_i'(\xi)]^2}{P_i(\xi) \cdot [1 - P_i(\xi)]}$$

  satisfies

$$I_{Y_i}(\xi) \;=\; \alpha_i^2 \cdot\; P_i(\xi) \cdot\; [1 - P_i(\xi)], \qquad i = 1, \ldots, m.$$

- In the 2-PL-model, the conditional variance $Var(Y_i \mid \xi)$ is *not* identical to the information function, if $\alpha_i \neq 1$, that is,

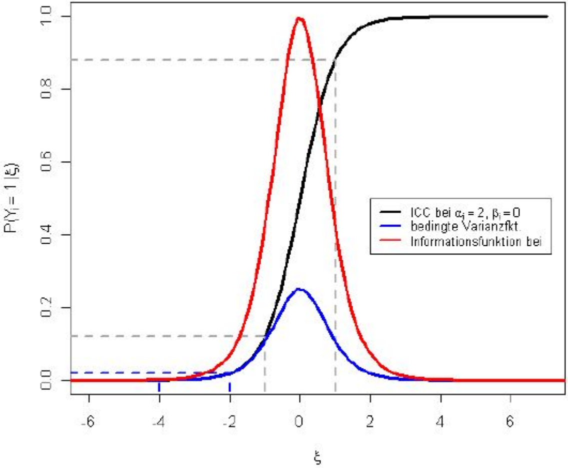
$$\alpha_i \neq 1 \;\;\Rightarrow\;\; Var(Y_i \mid \xi) \neq I_{Y_i}(\xi).$$

  The term $P_i'(\xi)$ denotes the first derivative of $P_i(\xi)$ with respect to $\xi$.
- Test information function:

$$I_{Y_1,\ldots,Y_m}(\xi) = \sum_{i=1}^{m} I_{Y_i}(\xi)$$

# 2-PL-Birnbaum model

- ICC and Information function




$\alpha_i = 2$

$\beta_i = 0$

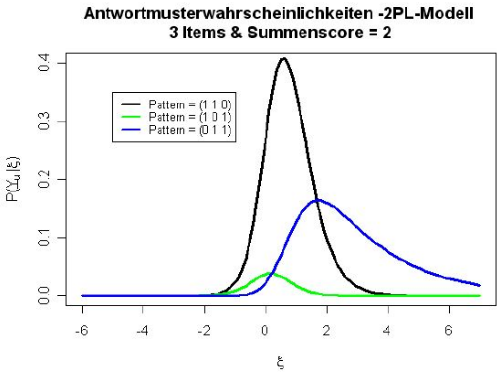
# 2-PL-Birnbaum model

- The asymptotic standard errors are again (as in the Rasch model):

$$
\begin{aligned}
SE_{\hat{\xi}}(\xi) &\approx \sqrt{\frac{1}{I_{Y_1,\ldots,Y_m}(\xi)}} \\
&= \frac{1}{\sqrt{\sum_{i=1}^{m} I_{Y_i}(\xi)}} \\
&= \frac{1}{\sqrt{\sum_{i=1}^{m} \alpha_i^2 \cdot P_i(\xi) \cdot [1 - P_i(\xi)]}}
\end{aligned}
$$

# Parameter estimation

- Maximum likelihood estimation:




3 items:

$$\alpha_1 = 0.5, \beta_1 = -1$$
$$\alpha_2 = 3, \beta_2 = 0$$
$$\alpha_3 = 0.5, \beta_3 = 1$$

# Parameter estimation

- Maximum likelihood estimation:
  - Sum score is not a sufficient statistic for the estimation of the person parameter
  - now it is important, *which* items are solved.
  - Persons with identical sum scores, but different response patterns can now have different estimates of their person parameters as well as different standard errors
  - Winmira cannot be used for the 2PL-Birnbaum model

# 2-PL-Birnbaum model – Mplus

- Model specification in Mplus:

Mplus_16items_2p.inp

```
TITLE:      16 Items 1 latente Dimension 2PL;

DATA:       FILE IS 16items_2pl_1latent.dat;
            TYPE IS INDIVIDUAL;

VARIABLE:   NAMES ARE i1-i16;
            USEVARIABLES ARE i1-i16;

            CATEGORICAL ARE i1-i16;

ANALYSIS:   Estimator=MLR;

MODEL:      XI BY i1-i16*;

            XI@1;
            [XI@0];

OUTPUT:     TECH1;
SAVE:       FILE IS person_est.dat;
            SAVE=FSCORES;
```

Logistisches IRT-Modell

BY – Statement für
Itemdiskrimintionen

Speichern der Personenparameter
(EAP)

# 2-PL-Birnbaum model – Mplus

- Output:
  factor analytic parametrization: $logit_i = \alpha_i \cdot \xi - \gamma_i$

MODEL RESULTS

| | Estimate | S.E. | Est./S.E. | Two-Tailed P-Value |
|---|---|---|---|---|
| XI BY | | | | |
| Y0 | 3.868 | 1.371 | 2.822 | 0.005 |
| Y1 | 1.903 | 0.277 | 6.861 | 0.000 |
| Y2 | 1.110 | 0.133 | 8.362 | 0.000 |
| Y3 | 1.205 | 0.141 | 8.552 | 0.000 |
| ... | | | | |
| Y10 | 3.187 | 0.833 | 3.824 | 0.000 |
| Means | | | | |
| XI | 0.000 | 0.000 | 999.000 | 999.000 |
| Thresholds | | | | |
| Y0$1 | -8.729 | 2.512 | -3.475 | 0.001 |
| Y1$1 | -3.598 | 0.317 | -11.338 | 0.000 |
| Y2$1 | -1.390 | 0.103 | -13.452 | 0.000 |
| Y3$1 | -1.047 | 0.097 | -10.808 | 0.000 |
| ... | | | | |
| Y10$1 | 7.623 | 1.482 | 5.142 | 0.000 |
| Variances | | | | |
| XI | 1.000 | 0.000 | 999.000 | 999.000 |

Diskriminationsparameter $(\alpha_i)$
in der logistischen Metrik

transformierte
Itemschwierigkeitsparameter $(\gamma_i)$

# 2-PL-Birnbaum model – Mplus

- Output(continued):
  2-PL-Birnbaum model parametrization: $logit_i = \alpha_i \cdot (\xi^* - \beta_i)$

IRT PARAMETERIZATION IN TWO-PARAMETER LOGISTIC METRIC
WHERE THE LOGIT IS DISCRIMINATION*(THETA - DIFFICULTY)

Item Discriminations

| | | | | |
|---|---|---|---|---|
| XI BY | | | | |
| Y0 | 3.868 | 1.371 | 2.822 | 0.005 |
| Y1 | 1.903 | 0.277 | 6.861 | 0.000 |
| Y2 | 1.110 | 0.133 | 8.362 | 0.000 |
| Y3 | 1.205 | 0.141 | 8.552 | 0.000 |
| ... | | | | |
| Y10 | 3.187 | 0.833 | 3.824 | 0.000 |

Diskriminationsparameter ($\alpha_i$) in der logistischen Metrik

Means

| | | | | |
|---|---|---|---|---|
| XI | 0.000 | 0.000 | 0.000 | 1.000 |

Item Difficulties

| | | | | |
|---|---|---|---|---|
| Y0$1 | -2.257 | 0.187 | -12.047 | 0.000 |
| Y1$1 | -1.891 | 0.153 | -12.396 | 0.000 |
| Y2$1 | -1.253 | 0.130 | -9.663 | 0.000 |
| Y3$1 | -0.869 | 0.097 | -8.948 | 0.000 |
| ... | | | | |
| Y10$1 | 2.392 | 0.204 | 11.711 | 0.000 |

Itemschwierigkeitsparameter ($\beta_i$ )

Variances

| | | | | |
|---|---|---|---|---|
| XI | 1.000 | 0.000 | 0.000 | 1.000 |

# 2PL-Birnbaum model

- **Uniqueness**
  The item parameters $\alpha_i$, $\beta_i$, and the latent variable $\xi$ are not uniquely defined by the assumptions of the 2PL-Birnbaum model. This is easily seen from

$$\alpha_i \cdot (\xi - \beta_i) \;=\; \frac{\alpha_i}{a} \cdot \big[(a \cdot \xi + b) - (a \cdot \beta_i + b)\big] \;=\; \alpha_i^* \cdot (\xi^* - \beta_i^*), \quad a, b \in \mathbb{R},$$

  with $\alpha_i^* := \dfrac{\alpha_i}{a}$, $\xi^* := a \cdot \xi + b$, and $\beta_i^* := a \cdot \beta_i + b$.
- **Admissible transformations**
  The item parameters $\beta_i$ and the latent variable $\xi$ have an **interval scale**. Admissible transformations are the linear transformations. The discrimination parameters $\alpha_i$ have a **ratio scale**. Admissible transformations are multiplications by a real number $\left(\dfrac{1}{a}\right)$.
- **Meaningful propositions, that is, propositions with invariant truth values**:
  - about ratios of differences between values of the latent variable $\xi$
  - about ratios of differences between item parameters $\beta_i$
  - about ratios of the discrimination parameters $\alpha_i$

# 2PL-Birnbaum model

**Scale fixing via**:

- Fixing via item parameters. For example, $\beta_1 = 0$ and $\alpha_1 = 1$. In this case, the latent variable $\xi$ is defined by

$$\xi := logit_1 = \ln\left(\frac{P(Y_1 = 1 \mid U)}{1 - P(Y_1 = 1 \mid U)}\right).$$

- Fixing the expectation and the variance of the latent variable. For example, $E(\xi) = 0$ and $Var(\xi) = 1$.

In both cases, the logit of each item is a (deterministic) linear function of the latent variable, and vice versa, the latent variable is a (deterministic) linear function of the logit of each item.