Lecture „Item Response Theory“

Link functions

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- In logit and probit models, a link function $f :]0,1[\to \mathbb{R}$ transforms the conditional probability $P(Y_i = 1 \mid U)$ (whose values are between 0 and 1, exclusively) to an unbounded real-valued random variable $f[P(Y_i = 1 \mid U)]$.

- A response function f^{-1} is the inverse function of the corresponding link function, which transforms $f[P(Y_i = 1 \mid U)]$ back to the probability scale. That is,

$$f^{-1}\big(f[P(Y_i = 1 \mid U)]\big) = P(Y_i = 1 \mid U).$$

Logit vs. Probit Function

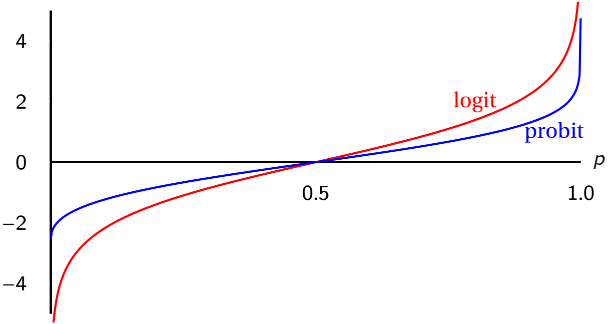
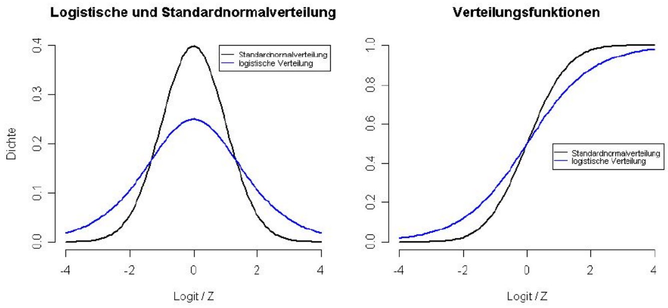


Abbildung: Graph of the logit and probit transformation of p

Logistic and standard normal distribution

- Comparing the logistic distribution and the normal distribution (density and distribution function).



Response functions and link functions in logit models and in probit models

Logit

- Link function: $logit_i := \ln\left(\frac{P(Y_i = 1 \mid U)}{P(Y_i = 0 \mid U)}\right)$
- Response function: $P(Y_i = 1 \mid U) = \frac{\exp(logit_i)}{[1 + \exp(logit_i)]}$

Probit

- Link function: $probit_i := \Phi^{-1}[P(Y_i = 1 \mid U)]$, where Φ^{-1} is the inverse function of the (cumulative) distribution function of the standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt, \quad \forall x \in \mathbb{R}.$$

- Response function $\Phi(probit_i)$

Logit model and probit model

In both models we assume: U-conditional independence of the items $Y_1, \ldots, Y_m$

$$\forall i \in I = \{1, \ldots, m\}: \quad P(Y_i = 1 \mid U, Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_m) = P(Y_i = 1 \mid U). \tag{1}$$

$$logit_i = \alpha_i^{log} \cdot (\xi^{log} - \beta_i^{log}), \quad \forall i = 1, \ldots, m$$

$$probit_i = \alpha_i^{pro} \cdot (\xi^{pro} - \beta_i^{pro}), \quad \forall i = 1, \ldots, m$$

Relationship between parameters of different models

- For the item discriminations, the following equations hold approximately:

$$\alpha_i^{log} \approx 1.7 \cdot \alpha_i^{pro}$$

- In both models the item difficulties are on the same scale as the latent variable

Probit models

- in probit models we can test model fit by a χ^2-test, and/or using the RMSEA

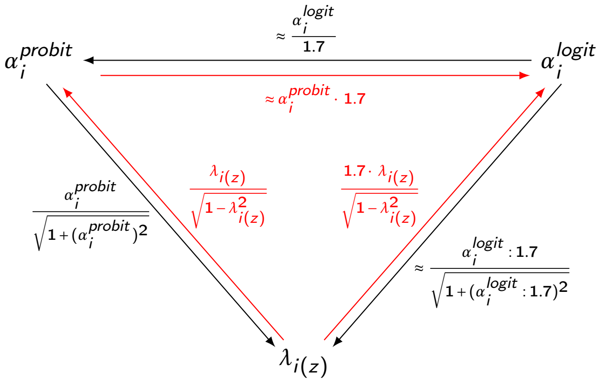
Factor analytic representation of the probit model for dichotomous variables

- Model equation and variance decomposition:

$$Y_i^* = \nu_i^* + \lambda_i^* \cdot \xi^* + \epsilon_i^*$$
$$Var(Y_i^*) = \lambda_i^{*2} \cdot Var(\xi^*) + Var(\epsilon_i^*)$$

Relationship between the parameters of different models

- Relationship between item discriminations/factor loadings, if we can assume: $E(\xi^{log}) = E(\xi^{pro}) = 0$ and $Var(\xi^{log}) = Var(\xi^{pro}) = 1$.



Birnbaum parametrization and factor analytic parametrization

- Also for item difficulties of the probit model, we can compute the intercept of the factor analytic representation of the probit model and vice versa

$$\beta_i^{pro} = \frac{\nu_i^*}{\lambda_i^*}$$

- and vice versa:

$$\nu_i^* = \beta_i^{pro} \cdot \lambda_i^*$$

Mplus syntax

- Example:

Mplus_16items_wlsmv.inp

```
TITLE:      16 Items 1 latente Dimension 2PL;

DATA:       FILE IS 16items_2pl_1latent.dat;
            TYPE IS INDIVIDUAL;

VARIABLE:   NAMES ARE i1-i16;
            USEVARIABLES ARE i1-i16;

            CATEGORICAL ARE i1-i16;

ANALYSIS:   Estimator=WLSMV;
            TYPE IS MEANSTRUCTURE;

MODEL:      XI BY i1-i16*;

            XI@1;
            [XI@0];


OUTPUT:     TECH1;
```

WLSMV = **W**eigthed **L**east **S**quare Estimator with **V**arianz and **M**ean adjusted χ^2-statistic