Chapter 3: Conditional Probability

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Examples

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Conditional Probability: Definition

Definition 3.1

If $\langle \Omega, \mathfrak{A}, P \rangle$ is a probability space, $A, B \in \mathfrak{A}$ and $P(B) > 0$, then

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

is called the conditional probability of the event A given the event B.

Conditional Probability: Venn-Diagram

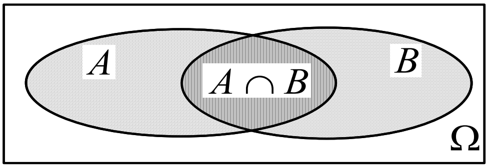

Figure 3.1. Illustration of a conditional probability by a Venn-diagram.

Conditional Probability Measure

Definition 3.2

If $\langle \Omega, \mathfrak{A}, P \rangle$ is a probability space, $A, B \in \mathfrak{A}$ and $P(B) > 0$, then we call the real function $P_B\colon \mathfrak{A} \to \mathbb{R}$ defined by

$$P_B(A) := P(A|B), \quad \text{for every } A \in \mathfrak{A},$$

the *B-conditional probability measure* on $\mathfrak{A}$ (with respect to P).

Independence of Events

Definition 3.3

Let $\langle \Omega, \mathfrak{A}, P \rangle$ be a probability space.

Two events $A, B \in \mathfrak{A}$ are called *independent* with respect to the probability measure P (short: *P-independent*), if

$$P(A \cap B) = P(A) \cdot P(B)$$

Independence of Events (continued)

Theorem 3.1

Let $\langle \Omega, \mathfrak{A}, P \rangle$ be a probability space. If $A, B \in \mathfrak{A}$ and $P(A) = 0$ or $P(A) = 1$, then the events A and B are independent.

Independence of Several Events

Definition 3.4

Let $\langle \Omega, \mathfrak{A}, P \rangle$ be a probability space. The events $A_1, ..., A_n \in \mathfrak{A}$ are called *independent* with respect to the probability measure P (short: *P-independent*) if and only if

$$P(A_{i_1} \cap \ldots \cap A_{i_m}) = P(A_{i_1}) \cdot \ldots \cdot P(A_{i_m})$$

for every $\{i_1, ..., i_m\} \subset \{1, \ldots, n\}$.

If this equation does not hold, the events $A_1, ..., A_n$ are called *stochastically dependent.*

Factorization Theorem

Theorem 3.2

Let $\langle \Omega, \mathfrak{A}, P \rangle$ be a probability space and $A_1, \ldots, A_n \in \mathfrak{A}$ events.

If $P(A_1 \cap \ldots \cap A_n) > 0$, then:

$$P(A_1 \cap \ldots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdot \ldots \cdot P(A_n|A_1 \cap \ldots \cap A_{n-1}).$$

Theorem of Total Probability

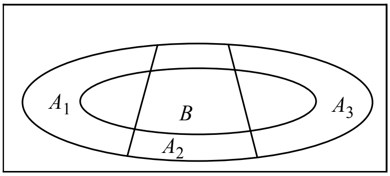

Figure 3.2. Venn-diagram illustrating the theorem of total probability.

Theorems of Total Probability and Bayes Theorem

Theorem 3.3

Let $\langle \Omega, \mathfrak{A}, P \rangle$ be a probability space and $A_1, \ldots, A_i, \ldots, A_n$ pairwise disjoint events with $P(A_i) > 0$, $i = 1, \ldots, n$. If $B \subset A_1 \cup \ldots \cup A_i \cup \ldots \cup A_n$.

Then:

(i) $$P(B) = P(B \cap A_1) + \ldots + P(B \cap A_i) + \ldots + P(B \cap A_n)$$

$$= P(B|A_1) \cdot P(A_1) + \ldots + P(B|A_i) \cdot P(A_i) + \ldots + P(B|A_n) \cdot P(A_n)$$

Theorem of Total Probability

and

(ii) $$P(A_i|B) = \frac{P(B|A_i) \cdot P(A_i)}{P(B|A_1) \cdot P(A_1) + \ldots + P(B|A_i) \cdot P(A_i) + \ldots + P(B|A_n) \cdot P(A_n)}.$$

Bayes Theorem