# Chapter 4: Random Variables



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## Informal Definitions



A *random variable* $X$ is a mapping $X\colon \Omega \to \Omega'$ which is uniquely defined by the answers to the two following questions:

- For which outcomes $\omega_i \in \Omega$ does $X$ have which values ( of the set $\Omega'$)?
- With which probability does $X$ take a value in a subset $A'$ of $\Omega'$?

The function assigning a probability for $X$ taking a value in a subset $A'$ of $\Omega'$ is called the *distribution* of the random variable $X$.

## Example 1



Single coin toss

$$X(\omega) = \begin{cases} 0, & \text{falls } \omega \in \{\omega_1, \omega_3, \omega_5\}, \\ 1, & \text{falls } \omega \in \{\omega_2, \omega_4, \omega_6\}. \end{cases}$$

## Example 2: Psychological Testing



Example for a random variable which is not real-valued

$\Omega = \Omega_U \times \Omega_O$

$U$: $\Omega \rightarrow \Omega'$, where $\Omega' = \Omega_U$ is the set of persons (the population) and

$U(\omega) = U(\langle u, o\rangle) = u \quad$ for all $\omega \in \Omega$.

## Random Variables: Formal Definition



**Definition 4.1**

Let $\langle \Omega, \mathfrak{A}, P \rangle$ be a probability space , $\Omega'$ a set and $\mathfrak{A}'$ a σ-algebra on $\Omega'$.

A mapping $X\colon \Omega \to \Omega'$ is called a *random variable*, if for the inverse image $X^{-1}(A')$ of every $A' \in \mathfrak{A}'$:

$$X^{-1}(A') \in \mathfrak{A}.$$

The set of all inverse images $X^{-1}(A')$ is called the "σ-*algebra generated by X*".

## Random Variables: Types



**Definition 4.2**

Let $X\colon \Omega \to \Omega'$ be a random variable on $\langle \Omega, \mathfrak{A}, P \rangle$.

(i) If the set $X(\Omega)$ of values of $X$ constists of countably infinite points, then $X$ is called *discrete*.

(ii) If $\Omega' \subset \mathbb{R}$, then $X$ is called *real* or *real-valued*.

(iii) If $\Omega' \subset \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$, then $X$ is called *numerical*.

(iv) If $\Omega' \subset \overline{\mathbb{R}}$ and the set $X(\Omega)$ of the values of $X$ is neither finite nor countably infinite, then $X$ is called *continuous*.

## Random Variables: Distributions



**Definition 4.3**

Let $\langle \Omega, \mathfrak{A}, P \rangle$ be a probability space, $X\colon \Omega \to \Omega'$ a random variable on $\langle \Omega, \mathfrak{A}, P \rangle$ and $\mathfrak{A}'$ a σ-algebra on $\Omega'$. Then we call the function $P^X\colon \mathfrak{A}' \to \mathbb{R}$ with

$$P^X(A') := P\,[X^{-1}(A')], \quad \text{for every } A' \in \mathfrak{A}',$$

the *distribution* of $X$ (with respect to $P$ ).

## Random Variables: Cumulative Distribution



**Definition 4.4**

Let $\langle \Omega, \mathfrak{A}, P \rangle$ be a probability space, $X\colon \Omega \to \overline{\mathbb{R}}$ a numerical random variable on $\langle \Omega, \mathfrak{A}, P \rangle$ and $\overline{\mathfrak{B}}$ the Borelian σ-algebra on $\overline{\mathbb{R}}$. Then the function $F^X\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ defined by

$$F^X(\alpha) := P^X([-\infty, \alpha]), \quad \alpha \in \overline{\mathbb{R}},$$

is called the *cumulative distribution* or *distribution function* of $X$ (with respect to $P$ ).

## Random Variables: Independence



**Definition 4.5**

Let $\langle \Omega, \mathfrak{A}, P \rangle$ be a probability space, $X$: $\Omega \rightarrow \Omega'_X$ und $Y$: $\Omega \rightarrow \Omega'_Y$ random variables, $\mathfrak{A}'_X$ a σ-algebra on $\Omega'_X$ and $\mathfrak{A}'_Y$ a σ-algebra on $\Omega'_Y$.

Then we call $X$ and $Y$ (stochastically) *independent* (with respect to $P$ ), if for every $(A', B') \in \mathfrak{A}'_X \times \mathfrak{A}'_Y$:

$$P\,[X^{-1}(A') \cap Y^{-1}(B')] = P\,[X^{-1}(A')] \cdot P\,[Y^{-1}(B')].$$

Otherwise $X$ and $Y$ are called (stochastically) *dependent*.