Expected Value: Definition

Definition 5.1

Let X be a random variable on the probability space $\langle \Omega, \mathfrak{A}, P \rangle$ with a finite number of values $x_1, ..., x_n$.

Then

$$E(X) := \sum_{i=1}^{n} x_i \cdot P(X = x_i)$$

is called *expected value* of X (with respect to P).

Expected Value: A Special Case

In the special case in which all values of X occur with the same probability $P(X = x_i) = 1/n$ (s. Def. 5.1), the equation simplifies to

$$E(X) := \frac{1}{n} \sum_{i=1}^{n} x_i$$

Example: Coin Toss

We consider a single dice toss with the set of outcomes $\Omega = \{\omega_1, ..., \omega_6\}$, where ω_i denotes the events, that the dice shows i points. Furthermore, let $\mathfrak{A}$ be the power set of Ω and the probability measure P: $\mathfrak{A} \to \mathbb{R}$ be defined by

$$P(\{\omega\}) := \frac{1}{6}, \quad \text{for every } \omega \in \Omega.$$

The real-valued random variable X: $\Omega \to \mathbb{R}$, is defined by

$$X(\omega_i) := i, \quad \text{for each } \omega_i \in \Omega.$$

X indicates the *number of points* that the dice shows.

X is a random variable on $\langle \Omega, \mathfrak{A}, P \rangle$ and for the expected value $E(X)$ of X the following equation holds:

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + ... + 6 \cdot \frac{1}{6} = (1 + 2 + ... + 6) \cdot \frac{1}{6} = 3.5.$$

Example: Expected Value of an Indicator Variable

Let Y be a random variable with only two values 0 and 1

(for example, odd and even number in a coin toss). Then the following equation holds

$$E(Y) = 0 \cdot P(Y = 0) + 1 \cdot P(Y = 1) = P(Y = 1).$$

The expected value of a random variable with two values 0 and 1, is equal to the probability $P(Y = 1)$ that Y takes the value 1.

Expected Value: Rules of Computation

If X and Y are numerical random variables on $\langle \Omega, \mathfrak{A}, P \rangle$ with finite expected values and α and $\beta \in \mathbb{R}$, then:

(i) $E(\alpha) = \alpha$

(ii) $E(\alpha \cdot X) = \alpha \cdot E(X)$

(iii) $E(\alpha \cdot X + \beta \cdot Y) = \alpha \cdot E(X) + \beta \cdot E(Y)$

Variance and Standard Deviation: Definition

Definition 5.2

Let X be a numerical random variable on the probability space $\langle \Omega, \mathfrak{A}, P \rangle$ with finite expected value. Then

$$Var(X) := E\,[[X - E(X)]^2]$$

is called the *variance* of X (with respect to P) and the positive square root

$$Std\,(X) = +\sqrt{Var(X)}$$

the *standard deviation* of X (with respect to P)

Variance: Rules of Computation

(i) $Var(X) = E(X^2) - E(X)^2$

(ii) $Var(X) = 0, \quad \text{if } X = \alpha$

If X and Y are numerical random variables on $\langle \Omega, \mathfrak{A}, P \rangle$ with finite expected values and α and $\beta \in \mathbb{R}$, then:

(iii) $Var(\alpha \cdot X) = \alpha^2 \cdot Var(X)$

(iv) $Var(\alpha + X) = Var(X)$

(v) $Var(\alpha \cdot X + \beta \cdot Y) = \alpha^2 \cdot Var(X) + \beta^2 \cdot Var(Y) + 2\alpha\beta \cdot Cov(X, Y)$

Variance of a Difference Variable

For the difference $X_1 - X_2$ of *uncorrelated* numerical random variables the following equations holds (Theorem (v))(with $\alpha_1 = 1$ and $\alpha_2 = -1$):

$$Var(X_1 - X_2) = Var(X_1) + Var(X_2), \quad \text{if } Cov(X_1, X_2) = 0.$$

The variance of a difference variable equals the *sum* of the variances, *if both variables are uncorrelated*.

Otherwise:

$$Var(X_1 - X_2) = Var(X_1) + Var(X_2) - 2\,Cov(X_1, X_2).$$

Covariance: Definition

Definition 5.3

Let X and Y be numerical random variables with finite expected values on the probability space $\langle \Omega, \mathfrak{A}, P \rangle$. Then we call

$$Cov(X, Y) := E[[X - E(X)] \cdot [Y - E(Y)]]$$

the *covariance* of X and Y (with respect to P).

Correlation as Expected Value of the Product of z-Transformed Variables

$$Kor(X, Y) := E\left[\frac{X - E(X)}{Std(X)} \cdot \frac{Y - E(Y)}{Std(Y)}\right]$$

If one or both standard deviations equals zero, then we define $Kor(X, Y) := 0$.

Correlation: Definition

Definition 5.4

Let X and Y be numerical random variables on the probability space $\langle \Omega, \mathfrak{A}, P \rangle$ with finite expected values and positive standard deviations.

Then

$$Kor(X, Y) = \frac{Cov(X, Y)}{Std(X) \cdot Std(Y)}$$

is called the *correlation* of X and Y (with respect to P). If at least one of the standard deviations is equal zero, then we define $Kor(X, Y) = 0$.

Lineare Quasi-Regression

Covariance and correlation characterize only the strength of a certain aspect of the stochastic dependency between two numerical random variables. The key point is, how strong the *linear dependency* between the two random variable is, i.e. a dependency that can be described as a straight line (a linear function of X) except for an error variable or residual ν

$$Y = \alpha_0 + \alpha_1 X + \nu$$

Both parameters α_0 and α_1 are uniquely defined, if the following equations for the error variable ν hold:

$$Cov(X, \nu) = 0 \quad \text{and} \quad E(\nu) = 0$$

Least-Squares-Criterion

An alternative and completly equivalent definition of the linear quasi-regression of Y on X, is to define it as the lineare function of X, that minimizes the following function of the real numbers a_0 and a_1:

$$LS(a_0, a_1) = E\left[(Y - (a_0 + a_1 \cdot X))^2\right]$$

Covariances: Rules of Computation I

Let X and Y be numerical random variables on $\langle \Omega, \mathfrak{A}, P \rangle$ with finite expected values and α and $\beta \in \mathbb{R}$, then:

(i) $Cov(X, Y) = E(X \cdot Y) - E(X) \cdot E(Y)$

(ii) $Cov(X, Y) = 0, \quad \text{if } X = \alpha$

(iii) $Cov(\alpha X, \beta Y) = \alpha \beta \, Cov(X, Y)$

(iv) $Cov(\alpha + X, \beta + Y) = Cov(X, Y)$

Covariances: Rules of Computation II

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Let X_1, X_2, Y_1 and Y_2 be numerical random variables on $\langle\Omega, \mathfrak{A}, P\rangle$ with finite expected values and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, then:

(v) $$Cov(\alpha_1 X_1 + \alpha_2 X_2, \beta_1 Y_1 + \beta_2 Y_2) = \alpha_1\ \beta_1\ Cov(X_1, Y_1)$$

$$+ \alpha_1\ \beta_2\ Cov(X_1, Y_2) + \alpha_2\ \beta_1\ Cov(X_2, Y_1) + \alpha_2\ \beta_2\ Cov(X_2, Y_2)$$

Correlations: Rules of Compuation I

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- $Kor(X, Y) = 0$, if $X = \alpha$
- $Kor(\alpha\, X, \beta\ Y) = Cor(X, Y)$
- $Kor(\alpha + X, \beta + Y) = Cor(X, Y)$

Numerical Example for the Calculation of a Covariance

X	Y	$P(X=x, Y=y)$	$X-E(X)$	$Y-E(Y)$	$[X-E(X)]\cdot[Y-E(Y)]$
0	0	5/40	–32/40	–15/40	48/160
0	1	3/40	–32/40	25/40	–80/160
1	0	20/40	8/40	–15/40	–12/160
1	1	12/40	8/40	25/40	20/160

$E(X) = 0 \cdot 8/40 + 1 \cdot 32/40 = 32/40$ $E(Y) = 0 \cdot 25/40 + 1 \cdot 15/40 = 15/40.$

$$Cov(X, Y) = E[[X-E(X)]\cdot[Y-E(Y)]]$$

$$= (48/160)\cdot 5/40 + (-80/160)\cdot 3/40 + (-12/160)\cdot 20/40 + (20/160)\cdot 12/40 = 0$$

The variables X and Y in this example have the covariance 0, i.e. they are uncorrelated.

Expected Value: General Definition

Definition 5.5

Let X be a numerical variable on the probability space $\langle \Omega, \mathfrak{A}, P\rangle$ and P^X its distribution. $E(X)$ is called the *expected value* of X (with respect to P), if

$$E(X) := \int x\, P^X(dx).$$