Contents

Stevens' Power Law I

- How does the length of the perceived line dependent on the length of the stimulus line?
- How does the perception error dependent on the length of the stimulus line?

Simple Lineare Regression: Definition

Definition 7.1

Let X and Y be numerical random variables of a common probability space, and both with a positive and finite variance. The regression $E(Y \mid X)$ of the regressand Y on the regressor X is called *linear in X*, if

$$E(Y \mid X) = \alpha_0 + \alpha_1 \cdot X,$$

where α_0 and α_1 are real numbers. The regressand Y is also called *linear regressivly dependent* from the regressor X.

Simple Linear Regression: Illustration

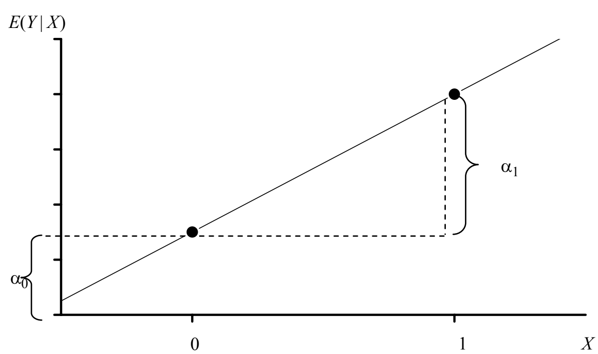

Figure 7.1 Note that α_0 is the intercept and α_1 is the slope of the straight line

Simple Lineare Regression: Residual

The variable Y is presented as the sum of a linear function of the variable X and the residual variable $\varepsilon = Y - E(Y \mid X)$:

$$Y = \alpha_0 + \alpha_1 \cdot X + \varepsilon.$$

Regressive Independence

A special case of linear regressive dependence is $\alpha_1 = 0$. We call this case *regressive independence*, because it implies:

$$E(Y \mid X) = E(Y) = \alpha_0.$$

Identifications of the Parameters

Note that:

$$\alpha_0 = E(Y) - \alpha_1 \cdot E(X),$$

and, if the variance of $Var(X) > 0$,

$$\alpha_1 = Cov(X, Y)/Var(X)$$

$$E(\varepsilon \mid X = x) = 0.$$

And furthermore

$$E(Y \mid X = x) = \alpha_0 + \alpha_1 \cdot x.$$

All the properties of the residual $\varepsilon := Y - E(Y \mid X)$ that were mentioned in the last chapter hold also here.

Simple Lineare Regression: Coefficient of Determination

In the case of a simple linear regression the coefficient of determination can be computed by the following equation:

$$R^2_{Y|X} = \alpha_1^2 \frac{Var(X)}{Var(Y)} = \frac{Cov(X,Y)^2}{Var(X) \cdot Var(Y)} = Kor(X,Y)^2 .$$

Parameterization of a Regression

There are many parameterizations for the same regression, i.e. a regression can be represented by many different equations.

Example: If you substitute the regressor X with $-X$, then for the slope coefficient results $-\alpha_1$ instead of α_1.

The main point is, that the regression $E(Y\,|\,X)$ doesn't change, i. e. $E(Y\,|\,X) = E(Y\,|\,-X)$

Dichotomous Regressor I

If the regressor X has only two real values, e.g. $X = 1$ for the experimental condition and $X = 0$ for the control condition, then $E(Y \mid X) = \alpha_0 + \alpha_1 \cdot X$ is *always* true. This is then called a *saturated model*. The two parameters in this case are:

$$\alpha_0 = E(Y \mid X = 0)$$

and

$$\alpha_1 = E(Y \mid X = 1) - E(Y \mid X = 0).$$

Dichotomer Regressor II

It is important to keep in mind that the interpretation of the two parameters depends on the coding of the values of X with 0 and 1. If you code the two conditions with –1 and +1, you have to interpret the two parameters differently:

$$\alpha_0 = [E(Y \mid X = 1) + E(Y \mid X = -1)] / 2$$

and

$$\alpha_1 = [E(Y \mid X = 1) - E(Y \mid X = -1)] / 2.$$

Stevens' Power Law II

If Stevens was right, then the relationship between the length of the presented line and the length of the produced by the person should be:

$$Y = b \cdot X^{a},$$

where a and b are real-valued. This seems to be straightforward for the described experiment. It is plausible that $Y = X$, i.e. $a = 1$ and $b = 1$.

A Stochastical Power Law

A first step to a more realistically formalization of Stevens' Power Law is to transform the two variables X and Y logarithmically. This leads to the linear equation:

$$ln(Y) = \alpha_0 + \alpha_1 \cdot ln(X)$$

where $\alpha_0 := ln(b)$ and $\alpha_1 := a$.

The stochastical character of the observed phenomenon can better be written

$$E[ln(Y) \mid ln(X)] = \alpha_0 + \alpha_1 \cdot ln(X)$$

A Stochastical Power Law

If we define the residual as

$$\varepsilon := ln(Y) - E[ln(Y) \mid ln(X)],$$

then we can write the equation as

$$ln(Y) = \alpha_0 + \alpha_1 \cdot ln(X) + \varepsilon.$$

The exponential tranformation leads to the stochastical power law

$$Y = b \cdot X^{a} \cdot \delta,$$

where $\delta := exp(\varepsilon)$. δ is a multiplicative error variable. The properties of δ can be derived from the properties of the residual ε. If the conditional distribution of ε given X is for example a normal distribution, then

$$E(\delta \mid X) = exp[(1/2) \cdot Var(\varepsilon \mid X)]$$

Logarithm

The *logarithm* to the base b, abbreviated as log_b, is defined as:
$log_b(x) := y$ only if $x = b^y$, where $x > 0$.

Rules of computation:

$log_b(x_1 \cdot x_2) = log_b(x_1) + log_b(x_2)$, where $b, x_1, x_2 > 0$ and $b \neq 1$

$log_b(1) = 0$

$log_b(\mathrm{b}) = 1$

$log_b(x_1 / x_2) = log_b(x_1) - log_b(x_2)$

$log_b(x^a) = a \cdot log_b(x)$.

And for the *natural logarithm*: $b = e$ and $ln(x) := log_e(x)$, where e is approximately 2.7183 the *Eulerian number*.

Exponential Function

The *exponential function* is the inverse function of the natural logarithm.

Note the rules of computation for the exponential function:

$exp[ln(x)] = x$, where $x > 0$

$exp(0) = 1$

$exp(x_1 + x_2) = exp(x_1) \cdot exp(x_2)$

$exp(x_1 - x_2) = exp(x_1) / exp(x_2)$

$exp(a \cdot x) = exp(x)^a$.