Part I: Elements of Probability Theory

- Chapter 2. Probability (to be treated today)
- Chapter 3: Conditional Probability
- Chapter 4: Random Variables
- Chapter 5: Expected value, variance, covariance and correlation

Chapter 2. Probability

Contents

- Examples
- Probability space
- Properties of probability

Examples

Example 1: Tossing a dice

$\Omega = \{\omega_1, ..., \omega_6\}$

Example 2: Double coin toss

$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$

Example 3: Psychological testing

$\Omega = \Omega_U \times \Omega_O$

Example 4: A typical psychological experiment

$\Omega = \Omega_U \times \Omega_X \times \Omega_Y$

Probability Space

A *probability space* represents the observed random experiment. It consists of three components

- a set of potential outcomes of the random experiment
- a set of potential events
- a probability measure

Random Experiment, Outcome, Event and Probability

A probability space $\langle \Omega, \mathfrak{A}, P \rangle$ represents the observed random experiment,

where:

- Ω is the *set of possible outcomes*
- $\mathfrak{A}$ is the *set of possible events* $A \subset \Omega$ and
- $P: \mathfrak{A} \rightarrow [0, 1]$ is a *probability measure on* $\mathfrak{A}$

The Set of Potential Outcomes

Example 1: Tossing a dice

$\Omega = \{\omega_1, ..., \omega_6\}$

Example 2: Double coin toss

$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$

Example 3: Psychological testing

$\Omega = \Omega_U \times \Omega_O$

The Set of Potential Events: Example 1

Potential events are always subsets of the set Ω of possible outcomes

Example 1 (Double coin toss):

$$\mathfrak{A} = \{\emptyset, \Omega\ \{(H, H)\}, \{(H, T), (T, H), (T, T)\}\}$$

The Set of Potential Events: Example 2

Double coin toss

The set of all subsets of Ω

$$\begin{aligned}\mathfrak{A} := \Big\{&\emptyset, \Omega, \{\langle T, T\rangle\}, \{\langle T, H\rangle\}, \{\langle H, T\rangle\}, \{\langle H, H\rangle\},\\ &\{\langle T, T\rangle, \langle T, H\rangle\}, \{\langle T, T\rangle, \langle H, T\rangle\}, \{\langle T, T\rangle, \langle H, H\rangle\},\\ &\{\langle T, H\rangle, \langle H, T\rangle\}, \{\langle T, H\rangle, \langle H, H\rangle\}, \{\langle H, T\rangle, \langle H, H\rangle\},\\ &\{\langle T, T\rangle, \langle T, H\rangle, \langle H, T\rangle\}, \{\langle T, T\rangle, \langle T, H\rangle, \langle H, H\rangle\},\\ &\{\langle T, T\rangle, \langle H, T\rangle, \langle H, H\rangle\}, \{\langle T, H\rangle, \langle H, T\rangle, \langle H, H\rangle\}\Big\}\end{aligned}$$

Some Potential Events in Psychological Testing Example

Table 2.1.
Some events and their mathematical representation

Event	Representation as a subset of $\Omega = \Omega_U \times \Omega_O$
Jim is drawn	$\{Jim\} \times \Omega_O$
Jim or John is drawn	$\{Jim, John\} \times \Omega_O$
The first problem is solved	$\Omega_U \times \{+\} \times \{+, -\}$
Jim is drawn and solves both problems	$\{\langle Jim, +, + \rangle\}$

Application and Interpretation Issues

- Problem: Probabilities are usually unkown
- Purpose: The estimation of the probabilities or other parameters
- Probabilities are *theoretical* quantities
- Example: A simple coin toss

Definition of a σ-Algebra

Definition 2.1.

Let $\mathfrak{A}$ be a set of subsets of a set Ω. Then $\mathfrak{A}$ is called a σ-*algebra* (or σ-*field*), if:

(a) $\Omega \in \mathfrak{A}$;

(b) if $A \in \mathfrak{A}$, then $\overline{A} \in \mathfrak{A}$ ($\overline{A}$ denotes the complement of A);

(c) if $A_1, A_2, \ldots$ is a sequence of elements of $\mathfrak{A}$, then the union $A_1 \cup A_2 \cup \ldots$ is an element of $\mathfrak{A}$.

Definition of Probability, etc. 1

Definition 2.2

Let $\mathfrak{A}$ be a σ-algebra with elements $A \subset \Omega$ and $P: \mathfrak{A} \rightarrow \mathbb{R}$ a function on $\mathfrak{A}$.

Now consider the conditions (the axioms of Kolmogorov):

(a) $P(A) \geq 0$, for every $A \in \mathfrak{A}$; *non-negativity*

(b) If $A_1, A_2, \ldots$ is a seqence of pairwise disjoint sets $A_i \in \mathfrak{A}$,

then: $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$ σ-*additivity*

(c) $P(\Omega) = 1$. *normalization*

Definition of Probability, etc. 2

If the conditons (a) to (c), the axioms of Kolmogorov hold, then we call:

(i) The function P *"probability measure",*

(ii) The tripel $\langle \Omega, \mathfrak{A}, P \rangle$ *"probability space",*

(iii) The elements $A_i \in \mathfrak{A}$ "*events",*

(iv) The value $P(A)$ "*probability of the event A"*,

(v) The sets $\{\omega\}$, $\omega \in \Omega$, "*elementary events"* and

(vi) The set Ω "*set of the possible events".*

A Complete Example of a Probability Space

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$$

Table 2.2.
All events and their probabilities of a double coin-toss

$A_i \in \mathfrak{A}$	$P(A_i)$	Note
$A_0 = \varnothing$	0	
$A_1 = \Omega$	1	condition (c) of Def. 2
$A_2 = \{\langle H, H\rangle\}$	1/4	fair coin!
$A_3 = \{\langle H, T\rangle\}$	1/4	dto.
$A_4 = \{\langle T, H\rangle\}$	1/4	dto.
$A_5 = \{\langle T, T\rangle\}$	1/4	dto.
$A_6 = \{\langle H, H\rangle, \langle H, T\rangle\}$	$P(A_2) + P(A_3) = 1/2$	condition (b) of Def. 2
$A_7 = \{\langle H, H\rangle, \langle T, H\rangle\}$	$P(A_2) + P(A_4) = 1/2$	dto.
$A_8 = \{\langle H, H\rangle, \langle T, T\rangle\}$	$P(A_2) + P(A_5) = 1/2$	dto.
$A_9 = \{\langle H, T\rangle, \langle T, H\rangle\}$	$P(A_3) + P(A_4) = 1/2$	dto.
$A_{10} = \{\langle H, T\rangle, \langle T, T\rangle\}$	$P(A_3) + P(A_5) = 1/2$	dto.
$A_{11} = \{\langle T, H\rangle, \langle T, T\rangle\}$	$P(A_4) + P(A_5) = 1/2$	dto.
$A_{12} = \{\langle H, H\rangle, \langle H, T\rangle, \langle T, H\rangle\}$	$P(A_2) + P(A_3) + P(A_4) = 3/4$	dto.
$A_{13} = \{\langle H, H\rangle, \langle H, T\rangle, \langle T, T\rangle\}$	$P(A_2) + P(A_3) + P(A_5) = 3/4$	dto.
$A_{14} = \{\langle H, H\rangle, \langle T, H\rangle, \langle T, T\rangle\}$	$P(A_2) + P(A_4) + P(A_5) = 3/4$	dto.
$A_{15} = \{\langle H, T\rangle, \langle T, H\rangle, \langle T, T\rangle\}$	$P(A_3) + P(A_4) + P(A_5) = 3/4$	dto.

Some Properties of Probability

Theorem 2.1

Let $\langle \Omega, \mathfrak{A}, P \rangle$ be a probability space and $A, B \in \mathfrak{A}$ events.

Then the following propositions hold:

(i) If $B \subset A$, then $P(A \setminus B) = P(A) - P(B)$ and $P(A) \geq P(B)$;

(ii) $P(A \setminus B) = P(A) - P(A \cap B)$;

(iii) If $\overline{A} := \Omega \setminus A$, then: $P(\overline{A}) = 1 - P(A)$;

(iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Some Properties of Probability (Diagram)

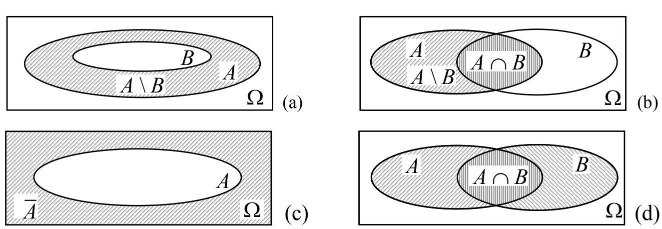

Figure 2.1

Venn-diagram illustrating the properties of a probability measure