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Example: The Ratio Model for Geometrical-Optical Illusions III

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How can we test the hypothesis of the ratio model for

$$E(\ln Y \mid \ln X, Z) = g_0(Z) + g_1(Z) \cdot \ln X$$

geometrical-optical illusions?

Before we can test this hypothesis we need the concepts of the *conditional linear quasi regression* and the *conditional nonlinear regression*.



Conditional Linear Quasi-Regression: Definition

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Definition 10.1. Let X , Y and Z be random variables on a common probability space and $g_0(Z)$, $g_1(Z)$ two numerical functions of Z . The function

$$Q(Y | X, Z) = g_0(Z) + g_1(Z) \cdot X$$

is called the *conditional linear quasi-regression* of Y on X with respect to Z , if for every value z of Z , the function

$$LS_z[f_0(z), f_1(z)] = E_{Z=z} [(Y - [f_0(z) + f_1(z) \cdot X])^2]$$

has its minimum for $f_0(z) = g_0(z)$ and $f_1(z) = g_1(z)$, where $g_0(z)$ and $g_1(z)$ are the values of $g_0(Z)$ and $g_1(Z)$ respectively.



Properties of the Residual

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The residual

$$\mathbf{n} := Y - g_0(Z) + g_1(Z) \cdot X$$

has the following properties:

$$E_{Z=z}(\mathbf{n}) = E(\mathbf{n} | Z = z) = 0,$$

$$Cov_{Z=z}(\mathbf{n}, X) = Cov(\mathbf{n}, X | Z = z) = 0,$$

$$E(\mathbf{n}) = 0, \text{ and } Cov(\mathbf{n}, X) = 0,$$

but, in general, not

$$E_{Z=z}(\mathbf{e} | X) = 0.$$



Parameterizations for the Conditional Linear Quasi-Regression

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All of the methods as discussed in the last chapter can be applied to the conditional linear quasi-regression in the same way.

Also identical is the identification for the coefficient of determination of the conditional linear quasi regression

$$Q_{Y|X,Z}^2 = \frac{\text{Var}[g_0(Z)] + \text{Var}[g_1(Z) \cdot X] + 2 \cdot \text{Cov}[g_0(Z), g_1(Z) \cdot X]}{\text{Var}(Y)}$$



Conditional Nonlinear Regression I

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Again we will consider indicator variables as a way to parameterize a conditional nonlinear regression. It is also called *means model*. For every combination of values of X and Z one indicator variable is formed:

$$I_{ij} = \begin{cases} 1, & \text{if } X = x_i \text{ and } Z = z_j \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, \dots, n, \quad j = 1, \dots, k.$$

The following parameterization as a means model provides a saturated parameterization:

$$E(Y|X, Z) = \mu_{11} \cdot I_{11} + \mu_{12} \cdot I_{12} + \dots + \mu_{ij} \cdot I_{ij} + \dots + \mu_{nk} \cdot I_{nk}.$$



Conditional Nonlinear Regression II

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Using the symbols μ_{ij} as coefficients of the I_{ij} makes sense, because they represent in fact the conditional expected values, as you can see in the following equations:

$$E(Y | X = x_1, Z = z_1) = \mu_{11} \cdot 1 + \mu_{21} \cdot 0 + \dots + \mu_{nk} \cdot 0 = \mu_{11},$$

$$E(Y | X = x_2, Z = z_1) = \mu_{11} \cdot 0 + \mu_{21} \cdot 1 + \dots + \mu_{nk} \cdot 0 = \mu_{21},$$

⋮

$$E(Y | X = x_n, Z = z_k) = \mu_{11} \cdot 0 + \mu_{21} \cdot 0 + \dots + \mu_{nk} \cdot 1 = \mu_{nk}.$$



Example: The Ratio-Model for Geometrical-Optical Illusions IV

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The basic statement of the ratio model for geometrical-optical illusions is that, for a constant context-stimulus ratio $Z = z$, the logarithm of the length of the line produced by the person ($\ln Y$) is *linearly regressively* dependent of the logarithm of the length of the stimulus line ($\ln X$). In other words, the conditional linear quasi-regression with respect to Z

$$\begin{aligned} Q(\ln Y | \ln X, Z) &= \beta_0 + \beta_1 I_{1/2} + \beta_2 I_{1/1} \\ &+ \gamma_0 \ln X + \gamma_1 I_{1/2} \ln X + \gamma_2 I_{1/1} \ln X. \end{aligned}$$

is equal to the real conditional linear regression with respect to Z .



Testing the Conditional Linear Regression

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The test is again done by comparing the estimates of the coefficients of determination of the conditional linear quasi-regression $\hat{Q}_{Y|X,Z}^2$ with the one of the saturated conditional nonlinear regression $\hat{R}_{Y|X,Z}^2$ by the following F -test:

$$F = \frac{(\hat{R}_{Y|X,Z}^2 - \hat{Q}_{Y|X,Z}^2)/(n-p)}{(1 - \hat{R}_{Y|X,Z}^2)/(N-n)}$$

where n is the number of parameters of the conditional nonlinear regression (the saturated model), p the number of parameters of the conditional linear quasi-regression and N the sample size.



Logistic Regression

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Logistic linear parameterization

$$P(Y = 1 | X, Z) = \frac{\exp[g_0(Z) + g_1(Z) \cdot X]}{1 + \exp[g_0(Z) + g_1(Z) \cdot X]}$$

Logistic polynomial parameterization

$$P(Y = 1 | X, Z) = \frac{\exp[g_0(Z) + g_1(Z) \cdot X + g_2(Z) \cdot X^2 + \dots]}{1 + \exp[g_0(Z) + g_1(Z) \cdot X + g_2(Z) \cdot X^2 + \dots]}$$

Saturated parameterization

$$P(Y = 1 | X, Z) = \frac{\exp(\lambda_{11}I_{11} + \lambda_{21}I_{21} + \dots + \lambda_{nk}I_{nk})}{1 + \exp(\lambda_{11}I_{11} + \lambda_{21}I_{21} + \dots + \lambda_{nk}I_{nk})}$$



Conditional Logistic Linear Regression: Figure

$P(Y=1 | X, Z)$

