Conditional Causal Effects

What you are going to learn

- An Introductory Example
- Basic Concepts
- Conditional Causal Effect
- Theorems
- Example II: Nonorthogonal Analysis of Variance

An Introductory Example

Table 17.1. Example in which each single person has a positive effect and the treatment assignment probability depends on gender.

Person	$P(U = u)$	$\mu_{1,u}$	$\mu_{2,u}$	$\mu_{1,u} - \mu_{2,u}$	Gender (Z)	$P(X = x_1 \mid U = u)$
u_1	1/6	80	68	12	z_1	3/4
u_2	1/6	93	81	12	z_1	3/4
u_3	1/6	103	89	14	z_1	3/4
u_4	1/6	116	102	14	z_1	3/4
u_5	1/6	132	123	9	z_2	1/4
u_6	1/6	148	137	11	z_2	1/4

Note. $Z = z_1$ denotes male, $Z = z_2$ denotes female.

Table 1. Fictitious example in which there is a positive individual causal effect for each subject and the probability to be assigned to the treatment condition depends on gender.

Subject	$P(U=u)$		$\mu_{1,u}$	$\mu_{2,u}$		$\mu_{1,u} - \mu_{2,u}$	Gender (Z)	$P(X=x_1 \mid U=u)$	$P(U=u \mid X=x_1)$	$P(U=u \mid X=x_2)$
u_1	1/6		80	68		12	z_1	3/4	3/14	1/10
u_2	1/6		93	81		12	z_1	3/4	3/14	1/10
u_3	1/6		103	89		14	z_1	3/4	3/14	1/10
u_4	1/6		116	102		14	z_1	3/4	3/14	1/10
		$CUE_{Z=z_1}(Y \mid X=x)$	98	85	$ACE_{Z=z_1}(1,2)$	13				
		$E_{Z=z_1}(Y \mid X=x)$	98	85	$PFE_{Z=z_1}(1,2)$	13				
u_5	1/6		132	123		9	z_2	1/4	1/14	3/10
u_6	1/6		148	137		11	z_2	1/4	1/14	3/10
		$CUE_{Z=z_2}(Y \mid X=x)$	140	130	$ACE_{Z=z_2}(1,2)$	10				
		$E_{Z=z_2}(Y \mid X=x)$	140	130	$PFE_{Z=z_2}(1,2)$	10				
		$CUE(Y \mid X=x)$	112	100	$ACE(1,2)$	12	$P(X=x_1)$	7/12		
		$E(Y \mid X=x)$	104	112	$PFE(1,2)$	−8				

Note: $CUE(Y \mid X=x)$ denotes the unbiased expected values given $X=x$ and $CUE_{Z=z}(Y \mid X=x)$ the $(Z=z)$-conditional causally unbiased expected values given $X=x$

The Underlying Random Experiment I

$$\Omega = \Omega_U \times \Omega_Z \times \Omega_X \times \Omega_Y \qquad \textit{set of possible outcomes}$$

Represents the random experiment "Draw a unit $u \in \Omega_U$, register its properties $\omega_Z \in \Omega_Z$ with respect to the conditioning variable (e.g. a pretest) as well as its assignment to a treatment condition $\omega_X \in \Omega_X$, and observe the value $\omega_Y \in \Omega_Y$ of the response variable Y."

$U: \Omega \rightarrow \Omega_U$ *observational-unit variable*

Its value is the drawn unit ("observational unit") u.

$Z: \Omega \rightarrow \Omega'_Z$

Its value represents for example the value of a possibly fallible pretest of a property of the observational unit (e.g. gender). Z may also be multidimensional and may represent with the two components (Z_1, Z_2) both at the same time.

The Underlying Random Experiment II

$X: \Omega \rightarrow \Omega'_X$ *treatment variable*

Its value represents the treatment condition ω_X , which the unit is assigned.

$Y: \Omega \rightarrow \mathbb{R}$ *response variable*

The observed response variable, whose dependency from X should be causally interpreted.

$E_{Z=z}(Y \mid X)$ *conditional treatment regression*

Describes the dependency within a value z of the control variable Z, which is to be causally interpreted.

$E(Y \mid X, U)$ *individual treatment regression*

Its values $\mu_{x\,u} := E(Y \mid X = x, U = u)$ are the individual expected values, i.e. the expected values of the response variable Y of a person u in a (treatment) condition $X = x$.

Conditional Causal Unbiasedness I

Definition 17.1. Let X: $\Omega \to \Omega'_X$ be a treatment variable, Y: $\Omega \to \mathbb{R}$ a response variable with positive and finite variance, Z: $\Omega \to \Omega'_Z$ a covariate and U: $\Omega \to \Omega_U$ the unit variable, all on a common probability space $\langle \Omega, \mathfrak{A}, P \rangle$, where Ω has the structure given in the last chapter. Further let $P(X = x, U = u) > 0$ and $P(X = x, Z = z) > 0$.

(i) The number

$$CUE_{Z=z}(Y \mid X = x) \quad := \textstyle\sum_u E_{Z=z}(Y \mid U = u, X = x)\, P_{Z=z}(U = u)$$

is called the $(Z = z)$-*conditional causally unbiased expected value of* Y given $X = x$.

Conditional Causal Unbiasedness II

(ii) The conditional regression $E_{Z=z}(Y \mid X)$ is called *causally unbiased*, if and only if for every value x of X

$$E_{Z=z}(Y \mid X = x) = CUE_{Z=z}(Y \mid X = x).$$

(iii) The $(Z = z)$-*conditional causal effect* of x_1 vs. x_2 on Y is defined:

$$ACE_{Z=z}(1,2) := CUE_{Z=z}(Y \mid X = x_1) - CUE_{Z=z}(Y \mid X = x_2).$$

Remark. The conditional average causal effect may also be computed by

$$\textstyle\sum_u [E_{Z=z}(Y \mid U = u,\ X = x_1) - E_{Z=z}(Y \mid U = u,\ X = x_2)]\, P_{Z=z}(U = u).$$

Theorems I

Theorem 17.1. Given the assumptions of Definition 17.1, stochastic independence of U and X given $Z = z$, i.e.

$$P_{Z=z}(U = u \mid X) = P_{Z=z}(U = u), \qquad \text{for each value } u \text{ of } U$$

implies causal unbiasedness of the regression $E_{Z=z}(Y \mid X)$.

Remark. $$P_{Z=z}(U = u \mid X) = P_{Z=z}(U = u), \qquad \text{for each value } u \text{ of } U$$

and

$$P_{Z=z}(X = x \mid U) = P_{Z=z}(X = x) \qquad \text{for each value } x \text{ of } X$$

are equivalent.

Theorem 17.2. Suppose the assumptions of Definition 17.1. hold.
If $E_{Z=z}(Y \mid X, U) = E_{Z=z}(Y \mid X)$, then the conditional regressions $E_{Z=z}(Y \mid X)$ are causally unbiased.

Theorems II

Theorem 17.3. Suppose the assumptions of Definition 17.1 hold. If X and U are $(Z = z)$-conditionally independent, or if $E_{Z=z}(Y \mid X, U) = E_{Z=z}(Y \mid X)$, then the conditional regression $E_{Z=z}(Y \mid X)$ are causally unbiased.

Theorem 17.4. Suppose the assumptions of Definition 17.1. hold. If

(a) For all values z of Z the conditional regression $E_{Z=z}(Y \mid X)$ is causally unbiased

and

(b) $E(Y \mid X, U, Z) = E(Y \mid X, U)$,

then

$$CUE(Y \mid X = x) = \sum_z E_{Z=z}(Y \mid X = x)\, P(Z = z), \qquad \text{for each value } x \text{ of } X.$$

Remarks I

(i) Provided that (a) and (b) hold, this theorem shows how to compute the causally unbiased conditional expected values $CUE(Y \mid X = x)$ from the conditional expected values $E_{Z=z}(Y \mid X = x)$ and the probabilities $P(Z = z)$. The difference $CUE(Y \mid X = x_1) - CUE(Y \mid X = x_2)$ of two such unbiased expected values yields the average causal effect $ACE(1, 2)$ in the total population, i.e.

$$ACE(1, 2) = CUE(Y \mid X = x_1) - CUE(Y \mid X = x_2)$$

(ii) Condition (a) is met, e.g., under the conditions mentioned in Theorems 1 to 3.

(iii) Condition (b) is met, e.g., if $Z = f(U)$, i.e., if Z is a (deterministic) function of U. Examples are Z_1 = gender, Z_2 = education and Z_3 = age.

Remarks II

(iv) Condition (b) is met also in other cases. If Z is for example a pre-test with a measurement error, then the true-score variable t_Z is responsible for the effect of Z, i.e.:

$$E(Y \mid X, U, Z) = E(Y \mid X, U, t_Z),$$

However, this implies:

$$E(Y \mid X, U, t_Z) = E(Y \mid X, U),$$

because the true-score-variable $t_Z := E(Y \mid U)$ is by definition a function of U. This yields condition (b). If Z is a fallible pretest, one can still create causal unbiasedness of the conditional regression $E_{Z=z}(Y \mid X)$, e.g. via conditional random assignment. In this way Theorem 4 will be applicable in practice.

Table 17.3. Example for a nonorthogonal design, in which there are prima facie effects of the therapy factor, although the corresponding average effects are zero.

	neediness			
	high	medium	low	
therapy	$Z = z_1$	$Z = z_2$	$Z = z_3$	total
1 $X = x_1$	120 (40)	110 (20)	60 (6)	(66)
2 $X = x_2$	100 (14)	100 (80)	100 (14)	(108)
3 $X = x_3$	80 (6)	90 (20)	140 (40)	(66)
total	(60)	(120)	(60)	(240)

Note. Presented are he expected values $E(Y \mid X = x,\ Z = z)$ of the response variable in the cells and in parentheses the number of observations.