Basic Concepts of Classical Test Theory: *Primitives* and *Theoretical Variables*

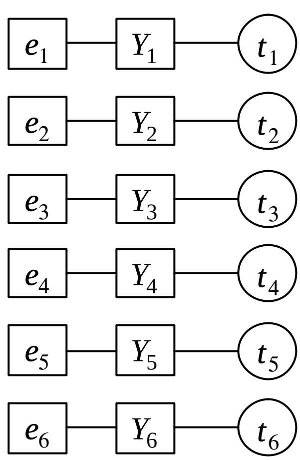
Primitives

the set of possible outcomes of the random experiment	$\Omega = \Omega_U \times \Omega_O$
test score variables	$Y_i: \Omega \rightarrow \mathrm{IR}$
person-projection	$U: \Omega \rightarrow \Omega_U$

Theoretical variables

true-score variables	$t_i := E(Y_i \mid U)$
measurement error variables	$e_i := Y_i - E(Y_i \mid U)$

Basic Concepts of Classical Test Theory: *Path Diagram*



Properties of *true-score* and *measurement error variables*

Properties of true-score variables and measurement error variables implied by their definition:

Decomposition of the variables $Y_i = t_i + e_i$ (1)

Decomposition of the variances $Var(Y_i) = Var(t_i) + Var(e_i)$ (2)

Other properties of true-score variables and measurement error variables implied by their definition

$Cov(t_i, e_j) = 0$ (3)

$E(e_i) = 0$ (4)

$E(e_i \mid U) = 0$ (5)

$E[e_i \mid f(U)] = 0$ (6)

for every mapping of U

Assumptions and *Models*

Assumptions

(a_1) t-equivalence $t_i = t_j$

(a_2) essential t-equivalence $t_i = t_j + \lambda_{ij}$, $\lambda_{ij} \in$ IR

(a_3) t-congenerity $t_i = \lambda_{ij0} + \lambda_{ij1} t_j$,

$\lambda_{ij0}, \lambda_{ij1} \in$ IR, $\lambda_{ij1} > 0$

(b) uncorrelated errors $Cov(e_i, e_j) = 0$, $i \neq j$

(c) equal error variance $Var(e_i) = Var(e_j)$

Models defined by these assumptions

- parallel tests: (a_1), (b) and (c)
- essential t-equivalent tests: (a_2) and (b)
- t-congeneric tests: (a_3) and (b)

Model of Parallel Tests:
Assumptions Defining the Model

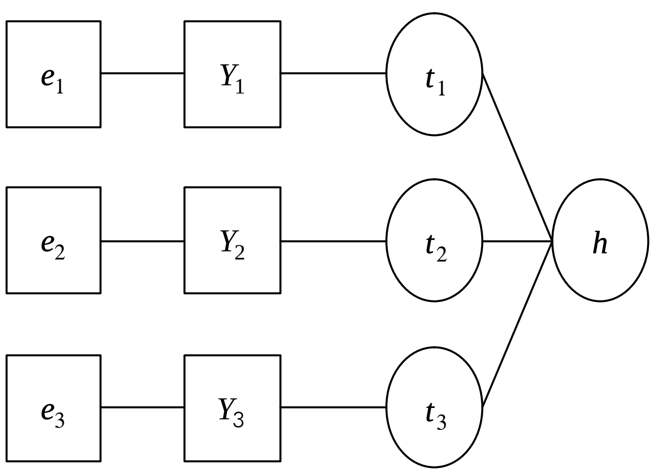
Definition: assumptions (a_1), (b) and (c)

(a_1) t-equivalence $t_i = t_j =: h$

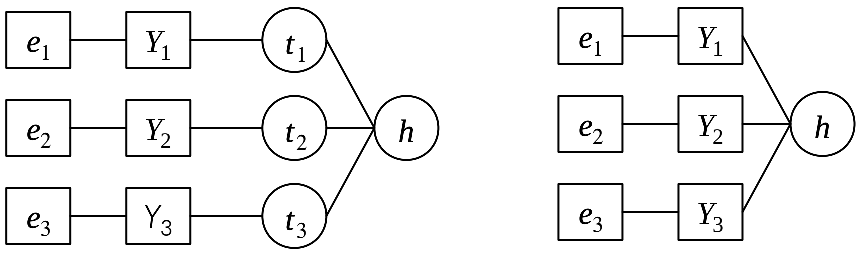
(b) uncorrelated errors $Cov(e_i, e_j) = 0, \quad i \neq j$

(c) equal error variances $Var(e_i) = Var(e_j)$

Model of Parallel Tests:
Path diagram I



Model of Parallel Tests: *Path diagram II*



Model of Parallel Tests: Implied *Covariance Structure*

$$
\begin{aligned}
Cov(Y_1, Y_2) &= Cov(h + e_1,\ h + e_2) \\
&= Cov(h, h) + Cov(h, e_2) + Cov(e_1, h) + Cov(e_1, e_2) \\
&= Var(h) =: \sigma_h^2
\end{aligned}
$$

$$
\begin{aligned}
Var(Y_i) &= Var(h + e_i) \\
&= Var(h) + Var(e_i) + 2 \cdot Cov(h, e_i) \\
&= \sigma_h^2 + \sigma_e^2
\end{aligned}
$$

$$
\begin{bmatrix}
\sigma_h^2 + \sigma_e^2 & \sigma_h^2 & \sigma_h^2 \\
\sigma_h^2 & \sigma_h^2 + \sigma_e^2 & \sigma_h^2 \\
\sigma_h^2 & \sigma_h^2 & \sigma_h^2 + \sigma_e^2
\end{bmatrix}
$$

Model of Parallel Tests: *Identification*

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$E(h) = E(Y_i)$

$Var(h) = Cov(Y_i, Y_j), \quad i \neq j$

$Var(e_i) = Var(Y_i) - Cov(Y_i, Y_j), \quad i \neq j$

$Rel(Y_i) = Kor(Y_i, Y_j), \quad i \neq j$

Model of Parallel Tests: *Testability*

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Testability in the total population

$E(Y_i) = \mu$

$Var(Y_i) = \sigma_Y^2$

$Cov(Y_i, Y_j) = \sigma_h^2$

Testability in each subpopulation *s* $\quad E^{(s)}(Y_i) = \mu^{(s)}$

The expected values in each subpopulation (*s*) are equal for every test score variable Y_i. Between different subpopulations, however, the expected values of the test score variables might differ.

Model of Essentially t-Equivalent Tests: *Assumptions Defining the Model*

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Definition: Assumptions (a_2) and (b)

(a_2) essential t-equivalence $t_i = t_j + \lambda_{ij}$

(b) uncorrelated errors $Cov(e_i, e_j) = 0, \quad i \neq j$

From (a_2) follows the existence of a latent variable h and constants λ_i with

$$t_i = h - \lambda_i$$

and

$$Y_i = h - \lambda_i + e_i \, .$$

Proof: Just set $j = 1$ and define $h := t_1$ and $\lambda_i := -\lambda_{i1}$.

Model of Essentially t-Equivalent Tests: : *Path Diagram*

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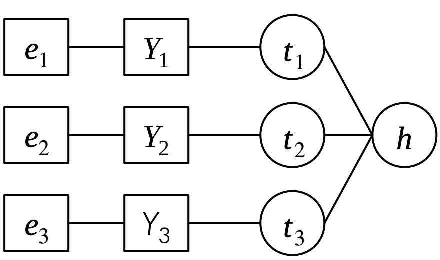

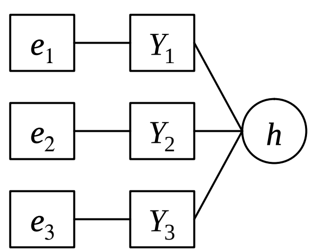

Model of Essentially t-Equivalent Tests: Implied *Covariance Structure*

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$$Cov(Y_1, Y_2) = Cov(h - \lambda_1 + e_1,\ h - \lambda_2 + e_2)$$
$$= Cov(h, h) + Cov(h, e_2) + Cov(e_1, h) + Cov(e_1, e_2)$$
$$= Var(h) =: \sigma_h^2$$

$$Var(Y_i) = Var(h - \lambda_i + e_i)$$
$$= Var(h) + Var(e_i) + 2 \cdot Cov(h, e_i)$$
$$= \sigma_h^2 + \sigma_e^2$$

$$\begin{bmatrix} \sigma_h^2 + \sigma_e^2 & \sigma_h^2 & \sigma_h^2 \\ \sigma_h^2 & \sigma_h^2 + \sigma_e^2 & \sigma_h^2 \\ \sigma_h^2 & \sigma_h^2 & \sigma_h^2 + \sigma_e^2 \end{bmatrix}$$

Model of Essentially t-Equivalent Tests: *Identification*

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$$Cov(Y_1, Y_2) = Cov(h - \lambda_1 + e_1, h - \lambda_2 + e_2)$$
$$= Cov(h, h) + Cov(h, e_2)$$
$$+ Cov(e_1, h) + Cov(e_1, e_2)$$
$$= Var(h) = \sigma_h^2$$

$$Var(Y_i) - Cov(Y_1, Y_2) = Var(e_i) = \sigma_{e_i}^2$$
$$Rel(Y_i) = Cov(Y_i, Y_j) \,/\, Var(Y_i), \quad i \neq j$$

Fixing of the location of h: $E(h) = 0$ implies

$$\lambda_i = -E(Y_i)$$

Model of Essentially *t-Equivalent* Tests: *Spearman-Brown Formula* and *Cronbach's a*

Assumptions (a_2) – (c) imply for *lengthened test* $S = Y_1 + \ldots + Y_m$:

Spearman-Brown-Formula $$Rel(S) = \frac{m \cdot Rel(Y_i)}{1 + (m-1) \cdot Rel(Y_i)}$$

Assumption (b = uncorrelated error variables) implies:

Cronbach's α $$a := \frac{m}{m+1}\left[1 - \frac{\sum_{i=1}^{m} Var(Y_i)}{Var(S)}\right]$$

is a lower bound of reliability of *S*.

Assumptions (a_2) – (c) imply: α is the reliability.

Model of Essentially *t*-Equivalent Tests: *Testability*

Testability in the total population $Cov(Y_i, Y_j) = \sigma_h^2,\ i \neq j,$

Testability in every subpopulation *s* $E^{(s)}(Y_i) - E^{(s)}(Y_j) = E^{(s)}(Y_i - Y_j) = \lambda_{ij}$

Same constant as in the total population*:* $E(Y_i) - E(Y_j) = E(Y_i - Y_j) = \lambda_{ij}$

Model of Essentially t-Equivalent Tests: *Uniqueness*

h and the coefficients λ_i uniquely defined up to *translations*.

Hence, h and λ_i are *difference scaled*.

Model of Essentialyl t-Equivalent Tests: *Meaningfulness*

Propositions about the following parameters are "meaningful", i.e., they have invariant truth values under translations (= the admissable transformations) of h and λ_i:

$$h(\omega_1) - h(\omega_2)$$

$$\lambda_i - \lambda_j$$

$$Var(h)$$

$$Rel(Y_i)$$

Model of Essentially t-Equivalent Tests: *An Example*

Table 11.1

Empirical covariances (lower triangular matrix), correlations (upper triangular matrix) and mean values of the state-anxiety-testhalfs of a total random sample

		occasion of measurement 1		occasion of measurement 2	
occasion of measurement		SA_1	SA_2	SA_3	SA_4
1	SA_1	24.67	*.88*	*.40*	*.44*
1	SA_2	21.90	25.14	*.41*	*.47*
2	SA_3	10.35	10.62	27.24	*.90*
2	SA_4	11.67	12.64	25.26	28.68
	mean	20.30	22.01	21.48	22.86

***notes*: *SA*:** State Anxiety. The first state anxiety test half at both occasions of measurement is equal the sum of the values of the items 2, 3, 4, 5, 7, 8, 12, 14, 16, 17, and the second is equal to the sum of the values of the items, 6, 9, 10, 11, 13, 15, 18, 19, 20

Model of Essentially t-Equivalent Tests: *Implied Covariances*

Table 11.2

Covariances implied by the model (lower triangular matrix) of the state anxiety test halves in the total sample

		occasion of measurement 1		occasion of measurement 2	
occasion of measurement		SA_1	SA_2	SA_3	SA_4
1	SA_1	24.67			
1	SA_2	15.41	25.14		
2	SA_3	15.41	15.41	27.24	
2	SA_4	15.41	15.41	15.41	28.68

Note: See Table 11.1 for the notation.

Model of Essentially t-Equivalent Tests: *Data for both Genders*

Covariances (lower triangular matrix), correlations (upper triangular matrix) and means (last row) of the state anxiety test halves of both gender groups

	males (N = 89)				**females** (N = 90)			
	SA_1	SA_2	SA_3	SA_4	SA_1	SA_2	SA_3	SA_4
SA_1	26.86	*.89*	*.52*	*.59*	22.77	*.88*	*.26*	*.28*
SA_2	22.78	24.44	*.55*	*.63*	21.22	25.81	*.27*	*.32*
SA_3	14.56	14.65	29.31	*.90*	6.35	7.00	25.31	*.91*
SA_4	16.58	17.14	26.63	29.92	6.97	8.51	24.02	27.66
	20.24	21.63	21.79	23.11	20.36	22.39	21.17	22.61

Notes: See Table 11.1 for notation.

SA_1 and SA_2 refer to the first occasion of measurement,

SA_3 and SA_4 refer to the second occasion of measurement.