Introduction to the Rasch-Model
Dichotomous Test Scores

True score variables of dichotomous test scores Y_i with the values 1 (e.g. for „solving ") and 0 (e.g. for „not solving" a task)

$$E(Y_i \mid U = u) = 1 \cdot P(Y_i = 1 \mid U = u) + 0 \cdot P(Y_i = 0 \mid U = u)$$
$$= P(Y_i = 1 \mid U = u)$$

The model of essentially *t*-equivalent variables:

$$P(Y_i = 1 \mid U) = h - \lambda_i$$

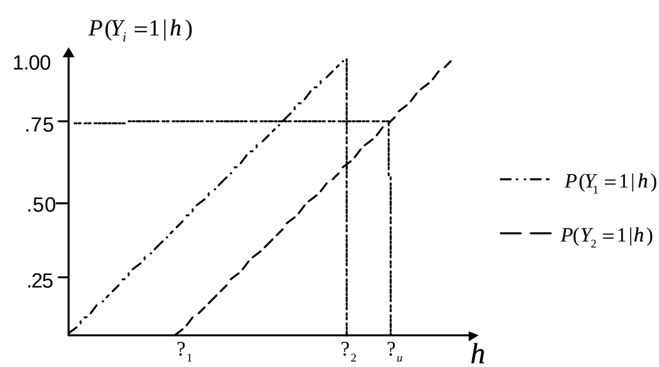
or

$$P(Y_i = 1 \mid h) = h - \lambda_i ,$$

because of

$$E[E(Y_i \mid U) \mid h] = E(Y_i \mid h) = P(Y_i = 1 \mid h)$$

Model of Essentially *t*-Equivalent Dichotomous Items



Odds ratio and Logits

Odds ratio

$$\frac{P(Y_i = 1 \mid U = u)}{1 - P(Y_i = 1 \mid U = u)} = \frac{P(Y_i = 1 \mid U = u)}{P(Y_i = 0 \mid U = u)}$$

Logarithm of the odds ratio (Logit)

$$\ln \frac{P(Y_i = 1 \mid U = u)}{P(Y_i = 0 \mid U = u)} = \ln P(Y_i = 1 \mid U = u) - \ln P(Y_i = 0 \mid U = u)$$

Logit variable

$$\ln \frac{P(Y_i = 1 \mid U)}{P(Y_i = 0 \mid U)} = \ln P(Y_i = 1 \mid U) - \ln P(Y_i = 0 \mid U)$$

Laws of Logarithm

1. $\log_b(a \cdot c) = \log_b a + \log_b c$ $\qquad \ln(a \cdot c) = \ln a + \ln c$

2. $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$ $\qquad \ln\left(\frac{a}{c}\right) = \ln a - \ln c$

3. $\log_b\left(a^n\right) = n \cdot \log_b a$ $\qquad \ln\left(a^n\right) = n \cdot \ln a$

4. $\log_b\left(\sqrt[n]{a}\right) = \frac{1}{n}\log_b a$ $\qquad \ln\left(\sqrt[n]{a}\right) = \frac{1}{n}\ln a$

Laws of Exponential Function

1. $\exp(a + c) = \exp(a) \cdot \exp(c)$
2. $\exp(a - c) = \frac{\exp(a)}{\exp(c)}$
3. $(\exp(a))^c = \exp(a \cdot c)$
4. $\ln(\exp(a)) = a$
5. $\exp(\ln(a)) = a$

Rasch-Homogeneity

First assumption: Rasch-Homogeneity

For each pair of logit variables there is a constant κ_{ij}, such that:

$\text{logit}_i = \text{logit}_j + \kappa_{ij}$.

Remember: $\text{logit}_i := \ln \frac{P(Y_i = 1 | U)}{P(Y_i = 0 | U)}$

This assumption is equivalent to:
for each variable Y_i, $i = 1, ..., m$, there is a random variable
? and a constant $?_i \in$ IR, for which the *fundamental law*

$$P(Y_i = 1 | U) = \frac{\exp(x - ?_i)}{1 + \exp(x - ?_i)}$$

holds.

Item Characteristic

$$P(Y_i = 1 \mid x) = \frac{\exp(x - ?_i)}{1 + \exp(x - ?_i)}$$

Illustration of the Item Characteristic

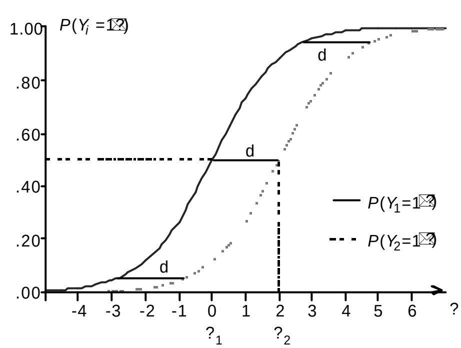


illustration of the item characteristics of two Rasch-homogeno

items Y_1 and Y_2 with the item coefficients $?_1 = 0$ and $?_2 = 2$.

d = $|\ ?_1 - ?_2\ |$.

Another Notation

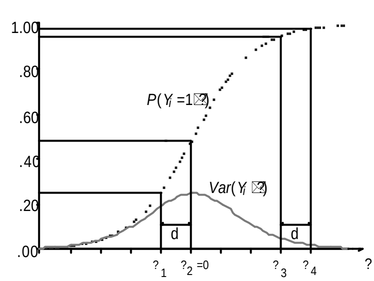
x_u denotes a value of the variable *?*. Therefore, the Rasch-Modell can be written as follows:

$$P(Y_i = 1 \mid U = u) = P(Y_i = 1 \mid x = x_u) = \frac{\exp(x_u - ?_i)}{1 + \exp(x_u - ?_i)}.$$

This notation means that the probability of solving item *i* of a given person *u* with ability x_u depends on the difference of the *coefficients* x_u and $?_i$, that is of the ability x_u and the difficulty $?_i$.

Conditional Variance and Information Function

The information function



item characteristic and *?*-conditional variance $Var(Y_i \mid ?) = Var(e_i \mid ?)$ of a stochastic variable Y_i with the item coefficient $?_i = 0$ in the Rasch-Model. The conditional variance is also the *information function* $I(Y_i \mid x = ?_u)$.

Theoretical Concepts

Admissible transformations and uniqueness

Transforming ? and all coefficients $?_i$ by adding the constant a:

$?' := ? + a$ resp. $?_i' := ?_i + a$,

the following equation holds:

$?' - ?_i' = (? + a) - (?_i + a) = ? - ?_i$.

Therefore, the next equation holds for the variables Y_i :

$$P(Y_i = 1 | U) = \frac{\exp(x' - ?_i')}{1 + \exp(x' - ?_i')}$$

Conditional Stochastic Independence

Second assumption: conditional stochastic independence

$$P(Y_i = 1 \mid U, Y_1, ..., Y_{i-1}, Y_{i+1}, ..., Y_m) = P(Y_i = 1 \mid U), \quad i = 1, ..., m$$

This equation is equivalent with

$$P(Y_{i_1} = 1, ..., Y_{i_n} = 1 \mid U) = P(Y_{i_1} = 1 \mid U) \cdot ... \cdot P(Y_{i_n} = 1 \mid U)$$

for every subset $\{i_1, \ldots, i_n\} \subset \{1, \ldots, m\}$.

Equality of Item Parameters and Invariance of Rank Orders

Equality of the item coefficients in subpopulations

$$?_i^{(s)} = ?_i^{(t)}$$

Equality of the rank order of the answering probability in subpopulations

$P(Y_i = 1) \leq P(Y_j = 1)$ if and only if

$P^{(s)}(Y_i = 1) \leq P^{(s)}(Y_j = 1)$

Equality of Probability Ratios

Equality of certain probability ratios in subpopulations

$$\frac{P^{(s)}(Y_i = 1, Y_j = 0)}{P^{(s)}(Y_i = 0, Y_j = 1)} = \frac{P^{(t)}(Y_i = 1, Y_j = 0)}{P^{(t)}(Y_i = 0, Y_j = 1)}$$

Estimation of the Theoretical Coefficients

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$$? _j - ?_i = \ln \left[\frac{P(Y_i = 1,\ Y_j = 0)}{P(Y_i = 0,\ Y_j = 1)} \right]$$

$$Rel(\hat{x}) = \frac{Var[E(\hat{x} \mid U)]}{Var(\hat{x})} = 1 - \frac{Var(e)}{Var(\hat{x})}$$

$$Var(\hat{x} \mid x = ?_u) \approx 1/[\sum_{i=1}^{m} I(Y_i \mid x = ?_u)]$$

$$\hat{x} \pm z_{a/2} \times \sqrt{[1/\sum_{i=1}^{m} I(Y_i \mid x = ?_u)]}$$

Models with Heterogeneous Item Discrimination and with Rating Probability

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models with heterogenous item discriminations

$$P(Y_i = 1 \mid U) = \frac{\exp[ß_i(x - ?_i)]}{1 + \exp[ß_i(x - ?_i)]}$$

models with guessing probability

$$P(Y_i = 1 \mid U) = ?_i + (1 - ?_i)\frac{\exp[ß_i(x - ?_i)]}{1 + \exp[ß_i(x - ?_i)]}$$

FPI-R Items 1 - 6

FPI-R-items of the Scale „life satisfaction“
(yes-no-answers, item number in parentheses)

1. Ich habe (hatte) einen Beruf, der mich voll befriedigt (3)
2. Ich lebe mit mir selbst in Frieden und ohne innere Konflikte (23)
3. Wenn ich noch einmal geboren würde, dann würde ich nicht anders leben wollen (29)
4. In meinem bisherigen Leben habe ich kaum das verwirklichen können, was in mir steckt (58)
5. Ich bin immer guter Laune (88)
6. Oft habe ich alles gründlich satt (94)

FPI-R Items 7 - 12

FPI-R-items of the Scale „life satisfaction“
(yes-no-answers, item number in parentheses)

7. Ich bin selten in bedrückter, unglücklicher Stimmung (100)
8. Ich grüble viel über mein bisheriges Leben nach (112)
9. Ich bin mit meinen gegenwärtigen Lebensbedingungen oft unzufrieden (119)
10. Alles in allem bin ich ausgesprochen zufrieden mit meinem bisherigen Leben (128)
11. Meine Partnerbeziehung (Ehe) ist gut (131)
12. Meistens blicke ich voller Zuversicht in die Zukunft (138)