Model of t-Congeneric Tests: *Assumptions Defining the Model*

Definition: assumptions (a_3) and (b)

(a_3) t-congenerity $\quad t_i = \lambda_{ij0} + \lambda_{ij1}\, t_j\,, \;\; \lambda_{ij0}, \lambda_{ij1} \in \mathrm{IR}, \;\; \lambda_{ij1} > 0$

(b) uncorrelated errors $\quad Cov(e_i, e_j) = 0, \quad i \neq j$

(a_3) implies the existence of a latent variable h and coefficients λ_{i1} and λ_i with:

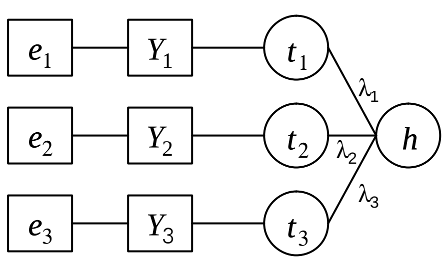
$$t_i = \lambda_{i1}(h - \lambda_i) \qquad \text{and} \qquad Y_i = \lambda_{i1}(h - \lambda_i) + e_i.$$

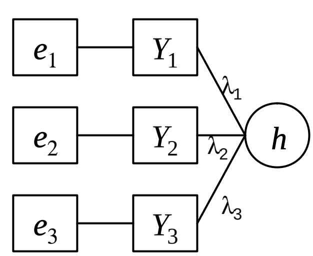
Proof: Just set $j = 1$ and define $h := t_1$, $\lambda_i := -\lambda_{i10}/\lambda_{i1}$ and $\lambda_{i1} := \lambda_{i11}$.

Classical parameterization:

$$Y_i = \lambda_{i0} + \lambda_{i1} h + e_i.$$

Model of t-Congeneric Tests: *Path Diagrams*





Model of t-Congeneric Tests:
Manifest and Latent Variables

Illustration of the relationship between manifest and latent variables in the model of t-congeneric tests

person	true scores		latent scores	manifest scores		error scores		
u	t_1	t_2	h	Y_1	Y_2	e_1	e_2	$P(Y_i = y_i \| U=u)$
				10	20	-2	-3	1/3
1	12	23	34	12	24	0	1	1/3
				14	25	2	2	1/3
				7	15	-3	-5	1/3
2	10	20	30	9	22	-1	2	1/3
				14	23	4	3	1/3
				3	14	-5	-3	1/3
3	8	17	26	10	15	2	-2	1/3
				11	22	3	5	1/3

Note: Fictitious numbers. Each of the three persons has his or her own (intra-individual) distribution of the Y-variable, but only one value on each latent variable t_1, t_2 and η.

Model of t-Congeneric Tests:
Uniqueness

(a_3) implies the existence of a latent variable h and coefficients λ_{i1} and λ_i with:

(1) $t_i = \lambda_{i1}(h - \lambda_i)$ and $Y_i = \lambda_{i1}(h - \lambda_i) + e_i$.

The latent variable h and the coefficients λ_{i1} and λ_i are *not uniquely* defined. They are uniquely defined only up to *linear transformations*, i.e. any $h' := \alpha + \beta h$ will also fulfil Equation 1, however, with new coefficients $?'_{i1} = \lambda_{i1}/\beta$ and $?'_i = \beta\lambda_i + \alpha$.

Proof: $t_i = (\lambda_{i1}/\beta) \cdot [(\alpha + \beta h) - (\beta\lambda_i + \alpha)]$.

Model of t-Congeneric Tests: *Identification*

In order to have a uniquely defined scale, we have to fix it.

Fixing the scale of h either by: $E(h) = 0$ and $Var(h) = 1$

or by: $\lambda_{10} = 0$ and $\lambda_{11} = 1$.

Model of t-Congeneric Tests: *Implied Covariance Structure*

$$Cov(Y_i, Y_j) = Cov(\lambda_{i0} + \lambda_{i1}h + e_i, \lambda_{i0} + \lambda_{j1}h + e_j)$$
$$= \lambda_{i1}\lambda_{j2}\, Var(h), \quad i \neq j$$

$$Var(Y_i) = Var(\lambda_{i0} + \lambda_{1}h + e_i)$$
$$= \lambda_{i1}^2\, Var(h) + Var(e_i)$$

$$\begin{bmatrix} \lambda_{11}^2 \sigma_h^2 + \sigma_{e_1}^2 & & \\ \lambda_{11}\lambda_{21}\sigma_h^2 & \lambda_{21}^2 \sigma_h^2 + \sigma_{e_2}^2 & \\ \lambda_{11}\lambda_{31}\sigma_h^2 & \lambda_{21}\lambda_{31}\sigma_h^2 & \lambda_{31}^2 \sigma_h^2 + \sigma_{e_3}^2 \end{bmatrix}$$

Model of t-Congeneric Tests: *Identification of the Loadings*

$$\frac{Cov(Y_i, Y_j)\ Cov(Y_i, Y_k)}{Cov(Y_j, Y_k)} = \frac{?_{i1}\ ?_{j1} Var(h)\ \cdot\ ?_{i1}\ ?_{k1} Var(h)}{?_{j1}\ ?_{k1} Var(h)}$$

$$= ?_{i1}^{2} Var(h), \quad i \neq j,\ i \neq k,\ j \neq k$$

Hence, if $Var(h) := 1$, then

$$?_{i1} = \sqrt{\frac{Cov(Y_i, Y_j)\ Cov(Y_i, Y_k)}{Cov(Y_j, Y_k)}}, \quad i \neq j,\ i \neq k,\ j \neq k$$

Model of t-Congeneric Tests: *Identification of other Parameters*

If we fix the scale of h by:

$E(h) = 0$ and $Var(h) = 1$,

then:

$Var(e_i) = Var(Y_i) - \lambda_{i1}^{2}$

$Rel(Y_i) = \lambda_{i1}^{2} / Var(Y_i)$

Model of t-Congeneric Tests: *Testability in the Total Population*

$$\frac{Cov(Y_i,Y_k)}{Cov(Y_j,Y_k)}=\frac{Cov(Y_i,Y_l)}{Cov(Y_j,Y_l)}$$

$$=\frac{?_{i1}\ ?_{k1}\,Var(h)}{?_{j1}\ ?_{k1}\,Var(h)}=\frac{?_{i1}\ ?_{l1}\,Var(h)}{?_{j1}\ ?_{l1}\,Var(h)}, \qquad i\neq k, i\neq l, j\neq k, j\neq l$$

Model of t-Congeneric Tests: *Testability Between Subpopulations*

$$\frac{E^{(1)}(Y_i)-E^{(2)}(Y_i)}{E^{(1)}(Y_j)-E^{(2)}(Y_j)}=\frac{E^{(3)}(Y_i)-E^{(4)}(Y_i)}{E^{(3)}(Y_j)-E^{(4)}(Y_j)}$$

Inserting themodel equation yields:

$$E^{(1)}(?_{i0}+?_{i1}h+e_i)=?_{i1}E^{(1)}(h).$$

Doing this for each of the eight expected values leads to:

$$\frac{?_{i1}E^{(1)}(h)-?_{i1}E^{(2)}(h)}{?_{j1}E^{(1)}(h)-?_{j1}E^{(2)}(h)}=\frac{?_{i1}E^{(3)}(h)-?_{i1}E^{(4)}(h)}{?_{j1}E^{(3)}(h)-?_{j1}E^{(4)}(h)}=\frac{?_{i1}}{?_{j1}}$$

Model of t-Congeneric Tests: *Meaningfulness*

Propositions about

$$\frac{h(\omega_1) - h(\omega_2)}{h(\omega_3) - h(\omega_4)} \quad \text{or} \quad \frac{\lambda_{i1}}{\lambda_{j1}}$$

and

$$\lambda_{i1}^2 \, Var(h) \,,$$

for instance, have *invariant truth values* under admissible transformations.

Model of t-Congeneric Tests: *An Example*

Tabelle 14.1

Empirical covariances (lower triangular matrix), correlations (upper triangular matrix) and means of the well-being tests in the total sample

		occasion of measurement 1		occasion of measurement 2	
occasion of measurement		WT_1	WT_2	WT_3	WT_4
1	WT_1	3.18	*.74*	*.64*	*.49*
1	WT_2	2.44	3.39	*.58*	*.66*
2	WT_3	1.89	1.78	2.78	*.68*
2	WT_4	1.42	1.96	1.83	2.63
	means	6.74	7.18	6.96	7.31

Notes: the tests WT_1 and WT_3 consist of the items 1, 11 and 20,

and the tests WT_2 and WT_4 consist of the items 2, 15 and 17 of the STAI

Model of t-Congeneric Tests : *Implied Covariances*

Tabelle 14.2

Covariances of the well-being tests in the total sample which are implied by the model (lower triangular matrix)

occasion of measurement		Occasion of measurement 1		Occasion of measurement 2	
		WT_1	WT_2	WT_3	WT_4
1	WT_1	3.17			
1	WT_2	2.18	3.36		
2	WT_3	1.86	2.10	2.78	
2	WT_4	1.71	1.93	1.65	2.65

Note: For notation see Table 14.1.